MATH 6280 - CLASS 19

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. The Whitehead theorem - Continued

Lemma 1.1. *Let $e : Y \to Z$ be an n–equivalence. Then given a diagram*

$$\begin{tikzcd}
S^{n-1} \arrow[rr] \arrow[rrrr, bend left, "g"] \arrow[dr, "i_1"'] & & CS^{n-1} \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& S^{n-1} \times I \arrow[rr] \arrow[drrr, bend right, "h"'] & & CS^{n-1} \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
S^{n-1} \arrow[rr] \arrow[ur, "i_0"] & & CS^{n-1} \arrow[rr, "f"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}$$

the dotted arrows exist.

Proof. Proof of Lemma 1.1. The proof is directly adapted from *Concise* (Ch. 9, §6).

First, we choose some points. Let $* \in S^{n-1}$ and $\bullet \in CS^{n-1}$ be the cone point. Define

$$\begin{aligned} y_1 &= g(*) \\ z_1 &= e \circ g(*) \\ z_0 &= f(*,0) = h(*,0) \\ z_{-1} &= f(\bullet). \end{aligned}$$

Recall that since e is an n–equivalence, $\pi_{n-1}(P_{e,z_1}, (y_1, c_{z_1})) = 0$. The idea is to construct a lift

$$\begin{tikzcd} & S^{n-1} \arrow[dl, dotted, "k_0"'] \arrow[d, "g"] & \\ P_{e,z_1} \arrow[r, "\pi_Y"'] & Y \arrow[r] & Z \end{tikzcd}$$

and use the provided null-homotopy of k_0 to get the desired extensions. Note that

$$P_{e,z_1} = \{(y,\alpha) \mid \alpha(0) = z_1,\ \alpha(1) = e(y)\} \subset Y \times Z^I.$$

The lift will have to satisfy

$$k_0(x) = (g(x), \gamma_x), \qquad \gamma_x(0) = z_1 = e \circ g(*) \qquad \gamma_x(1) = e \circ g(x)$$

So we need a path

$$z_1 = e \circ g(*) \overset{\gamma_x}{\rightsquigarrow} e \circ g(x)\ .$$

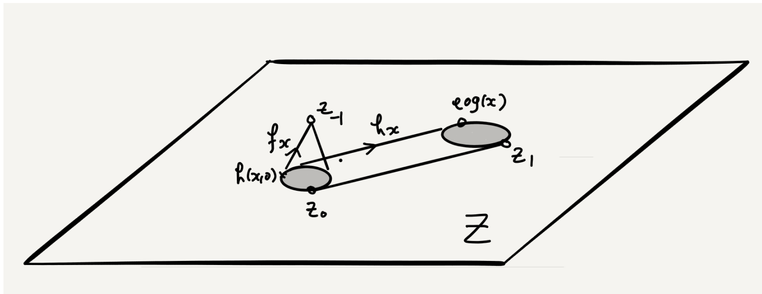
Fix $x \in S^{n-1}$. Let s be the cone coordinate of CS^{n-1} and t be the time coordinate of $S^{n-1} \times I$. There are various paths connecting the points z_{-1}, z_0, z_1.

$$h(x,0) = f(x,0) \overset{f_x(s)=f(x,s)}{\rightsquigarrow} z_{-1}$$

$$f(x,0) = h(x,0) \overset{h_x(t)=h(x,t)}{\rightsquigarrow} (e \circ g)(x)$$

In particular, after re-parametrizing, for any $x \in S^{n-1}$, we get a path

$$z_1 = (e \circ g)(*) \overset{\gamma_x = h_x \circ f_x^{-1} \circ f_* \circ h_*^{-1}}{\rightsquigarrow} (e \circ g)(x)$$



As discussed above, this gives a lift $k_0 : S^{n-1} \to P_{e,z_1}$ of g:

$$k_0(x) = (g(x), \gamma_x).$$

Now, $k_0 : S^{n-1} \to P_{e,z_1}$ is not a based map since

$$k_0(*) = (g(*), h_* \circ f_*^{-1} \circ f_* \circ h_*^{-1})$$

and $h_* \circ f_*^{-1} \circ f_* \circ h_*^{-1}$ is a non-constant path from z_1 to z_1. However, there is a path in P_{e,z_1} from

$$(g(*), h_* \circ f_*^{-1} \circ f_* \circ h_*^{-1}) \to (g(*), c_{z_1})$$

so there is a homotopy:

$$H_0 : S^{n-1} \times I \to P_{e,z_1}$$

from k_0 to a based map $k_1 : S^{n-1} \to P_{e,z_1}$. Since $\pi_{n-1}(P_{e,z_1}, (y_1, c_{z_1})) = 0$, there is a homotopy

$$H_1 : S^{n-1} \times I \to P_{e,z_1}$$

from k_1 to the constant map at $(g(*), c_{z_1})$. Concatenating H_0 and H_1 and reparametrizing, we get an unbased homotopy

$$k : S^{n-1} \times I \to P_{e,z_1}$$

from k_0 to the constant map at $(g(*), c_{z_1})$.

Now,

$$k(x,t) = (\widetilde{g}(x,t), \alpha(x,t))$$

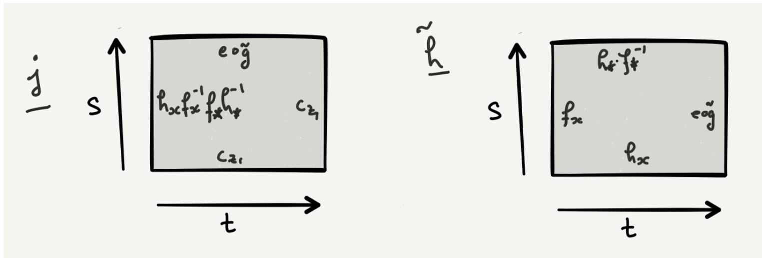
for some function $\widetilde{g} : S^{n-1} \times I \to Y$ and $\alpha : S^{n-1} \times I \to Z^I$

$$k(x,0) = (\widetilde{g}(x,0), \alpha(x,0)) = k_0(x) = (g(x), \gamma_x)$$

$$k(x,1) = (\widetilde{g}(x,1), \alpha(x,1)) = (g(*), c_{z_1}).$$

In particular, $\widetilde{g} : S^{n-1} \times \{1\} = g(*)$. So, we can extend $\widetilde{g} : CS^{n-1} \to Y$. That's our first lift!

We now need a map
$$\widetilde{h} : CS^{n-1} \times I \to Z$$
such that
$$\begin{aligned}
\widetilde{h}((x,s),1) &= e \circ \widetilde{g}(x,s) \\
\widetilde{h}((x,s),0) &= f(x,s) = f_x(s) \\
\widetilde{h}((x,0),t) &= h(x,t) = g_x(t).
\end{aligned}$$
Equivalently, swapping the time coordinates, we need a map
$$\widetilde{h} : S^{n-1} \times I \times I \to Z$$
such that
$$\begin{aligned}
\widetilde{h}(x,1,s) &= e \circ \widetilde{g}(x,s) \\
\widetilde{h}(x,0,s) &= f(x,s) = f_x(s) \\
\widetilde{h}(x,t,0) &= h(x,t) = h_x(t) \\
\widetilde{h}(x,t,1) &= \widetilde{h}(x',t,1), \qquad \forall x', x \in S^{n-1}.
\end{aligned}$$
Consider
$$j : S^{n-1} \times I \times I \to Z$$
given by
$$j(x,t,s) = \alpha(x,t)(s).$$
We have
$$\begin{aligned}
j(x,1,s) &= \alpha(x,1)(s) = c_{z_1}(s) \\
j(x,0,s) &= \alpha(x,0)(s) = \gamma_x(s) = h_x \circ f_x^{-1} \circ f_* \circ h_*^{-1}(s) \\
j(x,t,0) &= \alpha(x,t)(0) = c_{z_1}(t) \\
j(x,t,1) &= \alpha(x,t)(1) = e \circ \widetilde{g}(x,t).
\end{aligned}$$
Therefore, we can get $\widetilde{k}$ by precomposing j with a reparametrization of the boundary of $I \times I$.



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2. Corrections

(1) For a relative CW–complex (X, A), $X^0 = A \cup P$ where P is a discrete set of points.

(2) $\pi_n(X, A)$ is n–connected if $A \to X$ is an n–equivalence. This implies that $\pi_q(X, A) = 0$ for $1 \leq q \leq n$ and $\pi_0(A) \to \pi_0(X)$ is surjective.

3. Cellular Approximation

Theorem 3.1. *Let $f : (X, A) \to (Y, B)$ be any map between CW–complexes. Then f is homotopic relative to A to a cellular map. That is, there is a homotopy $H : X \times I \to Y$ which is constant on A such that $H(x, 0) = f(x)$ and $H(-, 1)|_{X^n} \subset Y^n$.*

Proof. We do induction over the skeletons, using HELP and the fact that $X^n \to X$ is n–connected.

Start with $X^0 = A \cup P$ for P some discrete set of points. For each $p \in P$, choose a path γ_p from $f(p)$ to $y_p \in Y^0$. Let $g_0 : X^0 \to Y$

$$g_0(x) = \begin{cases} x & x \in A \\ \gamma_p(1) & x \in P. \end{cases}$$

Assume that we have $g_n : X^n \to Y$ such that $g_n(X^n) \subset Y^n$ and that $h_n : X^n \times I \to Y$ is a homotopy between g_n and $f|_{X^n}$ relative to A.

We have that $i : Y^{n+1} \to Y$ is an $n + 1$–equivalence and (X^{n+1}, X^n) is a relative CW complex of dimension $n + 1$. So we can use HELP to construct g_{n+1} and h_{n+1} which extend g_n and h_n

respectively:

$$
\begin{tikzcd}
X^n \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "g_n"] & & X^{n+1} \arrow[rr, dotted, "g_{n+1}"] \arrow[dr, "i_1"'] & & Y^{n+1} \arrow[dd, "i"] \\
& X^n \times I \arrow[rr] \arrow[drrr, bend right, "h_n"'] & & X^{n+1} \times I \arrow[dr, dotted, "h_{n+1}"] & \\
X^n \arrow[rr] \arrow[ur, "i_0"] & & X^{n+1} \arrow[rr, "f|_{X^{n+1}}"] \arrow[ur, "i_0"] & & Y
\end{tikzcd}
$$

$\square$