MATH 6280 - CLASS 18

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. The Whitehead theorem

Recall:

Proposition 1.1 (HELP)**.** *Suppose that (X,A) is a relative CW–complex of dimension $\leq n$. Suppose that $e : Y \to Z$ is an n–equivalence. Given a diagram*

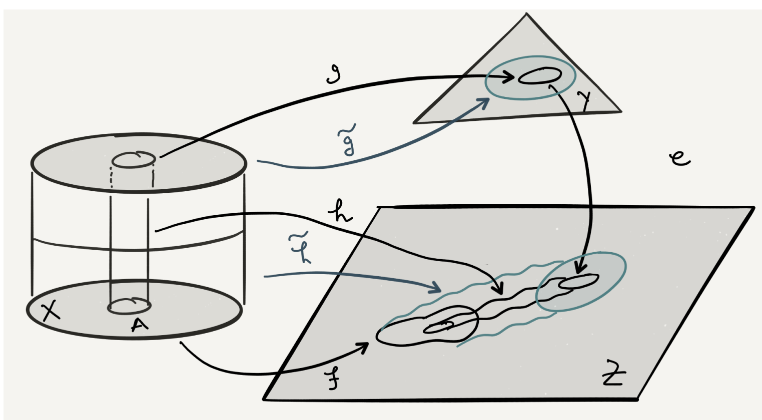
$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d, "e"] \\ X \arrow[r] \arrow[ur, dotted] & Z \end{tikzcd}$$

which commutes up to a homotopy H, there exists a lift $X \to Y$ which makes the upper triangle commute and makes the lower triangle commute up to a homotopy $\widetilde{H}$ that extends H.

In other words, in

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "g"] & & X \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& A \times I \arrow[rr] \arrow[drrr, bend right, "h"'] & & X \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
A \arrow[rr] \arrow[ur, "i_0"] & & X \arrow[rr, "f"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}$$

the dashed arrows exist.



Theorem 1.2 (Whitehead)**.** *(a) Let $e : Y \to Z$ be an n-equivalence. Let X be a CW-complex of dimension d. Then $[X, Y]_* \to [X, Z]_*$ is a bijection if $d < n$ and a surjection if $d = n$.*

(b) Let $e : Y \to Z$ be a weak equivalence and X be any CW-complex. Then $[X, Y]_* \to [X, Z]_*$ is a bijection.

(c) Let $e : Y \to Z$ be an n-equivalence. Suppose that Y and Z are CW complexes and $\dim Y, \dim Z < n$. Then e is a homotopy equivalence.

(d) Let $e : Y \to Z$ be a weak equivalence of CW–complexes. Then e is a homotopy equivalence.

Proof.

(a) Let $[f] \in [X, Z]_*$. Since $\dim(X, *) \leq n$, the diagram

$$
\begin{tikzcd}
* \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "*"] & & X \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& * \times I \arrow[rr] \arrow[drrr, bend right, "*"'] & & X \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
* \arrow[rr] \arrow[ur, "i_0"] & & X \arrow[rr, "f"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}
$$

gives a map $\widetilde{g} : X \to Y$ and a homotopy $\widetilde{h}$ between $e \circ \widetilde{g}$ and f. So $e_*([g]) = [e \circ g] = [f]$ and e_* is surjective.

Now, assume that X is a CW–complex of dimension $< n$. Suppose that $[e \circ g_0] = [e \circ g_1]$. Let $J = [0, 1]$ and choose a homotopy $h : X \times J \to Z$ from $e \circ g_0$ to $e \circ g_1$. Let

$$h : (X \times \{0\} \cup X \times \{1\}) \times I \to Z$$

be the constant map in t given by $h(p, t) = (e \circ g_0 \cup e \circ g_1)(p)$. Then the following diagram commutes:

$$
\begin{tikzcd}
(X \times \partial J) = X \times \{0\} \cup X \times \{1\} \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "g_0 \cup g_1"] & & X \times J \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& (X \times \partial J) \times I \arrow[rr] \arrow[drrr, bend right, "h"'] & & X \times J \times I \arrow[dr, dotted] & \\
(X \times \partial J) \arrow[rr] \arrow[ur, "i_0"] & & X \times J \arrow[rr, "H"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}
$$

Since $\dim X < n$, we have that $\dim(X \times J, X \times \partial J) \leq n$. Therefore, we get a lift

$$\widetilde{g} : X \times J \to Y$$

which is a homotopy between g_0 and g_1.

(b) For every n, we have that $[X_n, Y]_* \xrightarrow{e_*} [X_n, Z]_*$ is a bijection. Passing to the limit, we get that

$$[X, Y]_* = \lim_n [X_n, Y]_* \xrightarrow{e_*} \lim_n [X_n, Z]_* = [X, Z]_*$$

is a bijection,

(c) Consider $[Z, Y]_* \to [Z, Z]_*$. Since $\dim Z < n$, this is a bijection, so there is a map $f : Z \to Y$ such that $[e \circ f] = [\mathrm{id}_Z]$. That is, $Z \xrightarrow{f} Y \xrightarrow{e} Z$ is homotopic to the identity. Since, $\dim Y < n$, the map $e_* : [Y, Y]_* \to [Y, Z]_*$ is a bijection. However,

$$e_*[f \circ e] = [e \circ f \circ e] = [e] = [e \circ \mathrm{id}_Y] = e_*[\mathrm{id}_Y]$$

so $[f \circ e] = [\mathrm{id}_Y]$, that is, $Y \xrightarrow{e} Z \xrightarrow{f} Y$ is homotopic to the identity.

(d) Use (b) and the same proof as in (c). □

2. Proof of HELP

Lemma 2.1. *If $e : Y \to Z$ is an n–equivalence, then for any $z_0 \in e(Y)$, we have $\pi_q P_{e,z_0} = 0$ for $0 \leq q < n$ where*

$$P_{e,z_0} = \{(y, \alpha) \mid \alpha(0) = z_0, \alpha(1) = e(y)\}.$$

Proof. Choose y_0 such that $e(y_0) = z_0$. Let $p_0 = (y_0, c_{z_0}) \in P_{e,z_0}$ for c_{z_0} the constant path at z. Consider the long exact sequence on homotopy groups. We have

$$\ldots \to \pi_n(P_{e,z}, p_0) \to \pi_n(Y, y_0) \to \pi_n(Z, z_0) \to \pi_{n-1}(P_{e,z}, p_0) \to \pi_{n-1}(Y, y_0) \to \pi_{n-1}(Z, z_0) \to \ldots$$

Since $\pi_q(Y, y_0) \to \pi_q(Z, z_0)$ is an iso for $q < n$ and surjective for $q = n$, the conclusion follows. □

We will show:

Lemma 2.2. *Let $e : Y \to Z$ be such that $\pi_{n-1}(P_{e,f(y)}, (y, c_{f(y)})) = 0$ for all $y \in Y$. Then given a diagram*

$$
\begin{tikzcd}
S^{n-1} \arrow[rr] \arrow[rrrr, bend left, "g"] \arrow[dr, "i_1"'] & & CS^{n-1} \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& S^{n-1} \times I \arrow[rr] \arrow[drrr, bend right, "h"'] & & CS^{n-1} \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
S^{n-1} \arrow[rr] \arrow[ur, "i_0"] & & CS^{n-1} \arrow[rr, "f"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}
$$

the dotted arrows exist.

Lemma 2.3. *Let X be the pushout*

$$
\begin{tikzcd}
\sqcup_{i \in I} S_i^{q-1} \arrow[r, "\Phi"] \arrow[d, "\iota"] & A \arrow[d] \\
\sqcup_{i \in I} D_i^q \arrow[r, "\Psi"] & X.
\end{tikzcd}
$$

Then the pair (X, A) has HELP.

Proof. Let $n \geq q$ and $e : Y \to Z$ be an n–equivalence. Consider:

$$
\begin{tikzcd}
A \arrow[rr] \arrow[rrrr, bend left, "g"] \arrow[dr, "i_1"'] & & X \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& A \times I \arrow[rr] \arrow[drrr, bend right, "h"'] & & X \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
A \arrow[rr] \arrow[ur, "i_0"] & & X \arrow[rr, "f"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}
$$

By the previous lemma, we can find lifts

$$
\begin{tikzcd}
\sqcup_{i\in I} S_i^{q-1} \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "g\circ\Phi"] & & \sqcup_{i\in I} D_i^q \arrow[rr, dotted, "\widetilde{g}_0"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& \sqcup_{i\in I} S_i^{q-1}\times I \arrow[rr] \arrow[drrr, bend right, "h\circ(\Phi\times \mathrm{id}_I)"'] & & \sqcup_{i\in I} D_i^q\times I \arrow[dr, dotted, "\widetilde{h}_0"] & \\
\sqcup_{i\in I} S_i^{q-1} \arrow[rr] \arrow[ur, "i_0"] & & \sqcup_{i\in I} D_i^q \arrow[rr, "f\circ\Psi"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}
$$

Let $\widetilde{g}$ be the map given by

$$
\begin{tikzcd}
\sqcup_{i\in I} S_i^{q-1} \arrow[r, "\Phi"] \arrow[d, "\iota"] & A \arrow[d] \arrow[ddr, bend left, "g"] & \\
\sqcup_{i\in I} D_i^q \arrow[r, "\Psi"] \arrow[drr, bend right, "\widetilde{g}_0"'] & X \arrow[dr, dotted, "\widetilde{g}"] & \\
& & Y
\end{tikzcd}
$$

and $\widetilde{h}$ be the map given by

$$
\begin{tikzcd}
\sqcup_{i\in I} S_i^{q-1}\times I \arrow[r, "\Phi"] \arrow[d, "\iota"] & A\times I \arrow[d] \arrow[ddr, bend left, "h"] & \\
\sqcup_{i\in I} D_i^q\times I \arrow[r, "\Psi"] \arrow[drr, bend right, "\widetilde{h}_0"'] & X\times I \arrow[dr, dotted, "\widetilde{h}"] & \\
& & Z
\end{tikzcd}
$$

□

Lemma 2.4. *If $A\subset B\subset X$ and (B,A) and (X,B) have HELP, then so does (X,A).*

Proof. Exercise □

The proof of HELP now follows by induction over the skeleton of (X,A).