MATH 6280 - CLASS 17

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. n–equivalences continued

Recall,

Definition 1.1. For X and Y, $f : X \to Y$ is an n–equivalence if, **for all** $x \in X$

$$\pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for $0 \leq q < n$ and surjective if $q = n$.

Last time, we proved that

Proposition 1.2. *If $X \cup_\phi D^{n+1}$ for some map $\phi : S^n \to X$, then $(X \cup_\phi D^{n+1}, X)$ is n–connected.*

Corollary 1.3. *The pair (X, X^n) is n–connected. That is, $X^n \to X$ is an n–equivalence.*

Proof. Let $q \leq n$ and $f : (D^q, S^{q-1}) \to (X, X^n)$. Since $f(D^q)$ is compact, its image meets a finite number of cells. Therefore, it factors as

$$(D^q, S^{q-1}) \xrightarrow{f'} (Y, X^n) \xrightarrow{r} (X, X^n)$$

where $Y = X^n \cup_{f_1} D_1^{m_1+1} \cup_{f_2} D_2^{m_2+1} \cup \ldots \cup_{f_k} D_k^{m_k+1}$ for $n \leq m_i$. Now one applies (a) inductively to prove that f' is trivial $\pi_q(Y, X^n)$, and hence trivial in $\pi_q(X, X^n)$. $\square$

Remark 1.4. So to compute the first homotopy n groups of a CW–complexes, we can restrict our attention to its $n+1$–skeleton.

Example 1.5. $\pi_1 \mathbb{R}P^\infty = \mathbb{Z}/2$ and $\pi_k \mathbb{R}P^\infty = 0$ if $k \neq 1$. Indeed,

$$\pi_1 \mathbb{R}P^\infty \cong \pi_1 (\mathbb{R}P^\infty)^2 \cong \pi_1 \mathbb{R}P^2 \cong \mathbb{Z}/2.$$

Further,

$$\pi_k \mathbb{R}P^\infty \cong \pi_k (\mathbb{R}P^\infty)^{k+1} \cong \pi_k \mathbb{R}P^{k+1} \cong \pi_k S^{k+1} = 0.$$

2. The Whitehead theorem

The following proposition is called the *homotopy extension lifting property.* It is technical to state but will have important consequences.

Proposition 2.1 (HELP). *Suppose that (X, A) is a relative CW–complex of dimension $\leq n$. Suppose that $e : Y \to Z$ is an n–equivalence. Given a diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \arrow[d, "e"] \\
X \arrow[r] \arrow[ur, dotted] & Z
\end{tikzcd}$$

which commutes up to a homotopy H, there exists a lift $X \to Y$ which makes the upper triangle commute and makes the lower triangle commute up to a homotopy $\widetilde{H}$ that extends H.

In other words, in

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "g"] & & X \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& A \times I \arrow[rr] \arrow[drrr, bend right, "h"'] & & X \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
A \arrow[rr] \arrow[ur, "i_0"] & & X \arrow[rr, "f"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}$$

the dashed arrows exist.

Corollary 2.2. *If $A \to X$ is a relative CW–complex, then it is a cofibration.*

Proof. In

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "g"] & & X \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "\mathrm{id}_Y"] \\
& A \times I \arrow[rr] \arrow[drrr, bend right, "h"'] & & X \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
A \arrow[rr] \arrow[ur, "i_0"] & & X \arrow[rr] \arrow[ur, "i_0"] & & Y
\end{tikzcd}$$

the map $\widetilde{h}$ is the desired extension. ☐

Theorem 2.3 (Whitehead)**.** *(a) Let $e : Y \to Z$ be an n-equivalence. Let X be a CW-complex of dimension d. Then $[X, Y]_* \to [X, Z]_*$ is a bijection if $d < n$ and a surjection if $d = n$.*

(b) Let $e : Y \to Z$ be a weak equivalence and X be any CW-complex. Then $[X, Y]_* \to [X, Z]_*$ is a bijection.

(c) Let $e : Y \to Z$ be an n-equivalence. Suppose that Y and Z are CW complexes and $\dim Y, \dim Z < n$. Then e is a homotopy equivalence.

(d) Let $e : Y \to Z$ be a weak equivalence of CW–complexes. Then e is a homotopy equivalence.

Proof.

(a) Let $[f] \in [X, Z]_*$. Then the diagram

$$\begin{tikzcd}
* \arrow[rr] \arrow[dr, "i_1"'] \arrow[rrrr, bend left, "*"] & & X \arrow[rr, dotted, "\widetilde{g}"] \arrow[dr, "i_1"'] & & Y \arrow[dd, "e"] \\
& * \times I \arrow[rr] \arrow[drrr, bend right, "*"'] & & X \times I \arrow[dr, dotted, "\widetilde{h}"] & \\
* \arrow[rr] \arrow[ur, "i_0"] & & X \arrow[rr, "f"] \arrow[ur, "i_0"] & & Z
\end{tikzcd}$$

gives a map $\widetilde{g} : X \to Y$ and a homotopy $\widetilde{h}$ between $e \circ \widetilde{g}$ and f. So $e_*([g]) = [e \circ g] = [f]$ and e_* is surjective.

To be continued.... ☐