

## MATH 6280 - CLASS 17

### CONTENTS

1. $n$ -equivalences continued	1
2. The Whitehead theorem	2

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

### 1. $n$ -EQUIVALENCES CONTINUED

Recall,

**Definition 1.1.** For  $X$  and  $Y$ ,  $f : X \rightarrow Y$  is an  $n$ -equivalence if, **for all**  $x \in X$

$$\pi_n(X, x) \rightarrow \pi_n(Y, f(x))$$

is an isomorphism for  $0 \leq q < n$  and surjective if  $q = n$ .

Last time, we proved that

**Proposition 1.2.** *If  $X \cup_\phi D^{n+1}$  for some map  $\phi : S^n \rightarrow X$ , then  $(X \cup_\phi D^{n+1}, X)$  is  $n$ -connected.*

**Corollary 1.3.** *The pair  $(X, X^n)$  is  $n$ -connected. That is,  $X^n \rightarrow X$  is an  $n$ -equivalence.*

*Proof.* Let  $q \leq n$  and  $f : (D^q, S^{q-1}) \rightarrow (X, X^n)$ . Since  $f(D^q)$  is compact, its image meets a finite number of cells. Therefore, it factors as

$$(D^q, S^{q-1}) \xrightarrow{f'} (Y, X^n) \xrightarrow{r} (X, X^n)$$

where  $Y = X^n \cup_{f_1} D_1^{m_1+1} \cup_{f_2} D_2^{m_2+1} \cup \dots \cup_{f_k} D_k^{m_k+1}$  for  $n \leq m_i$ . Now one applies (a) inductively to prove that  $f'$  is trivial in  $\pi_q(Y, X^n)$ , and hence trivial in  $\pi_q(X, X^n)$ .  $\square$

**Remark 1.4.** So to compute the first homotopy  $n$  groups of a CW-complexes, we can restrict our attention to its  $n + 1$ -skeleton.

**Example 1.5.**  $\pi_1 \mathbb{R}P^\infty = \mathbb{Z}/2$  and  $\pi_k \mathbb{R}P^\infty = 0$  if  $k \neq 1$ . Indeed,

$$\pi_1 \mathbb{R}P^\infty \cong \pi_1 (\mathbb{R}P^\infty)^2 \cong \pi_1 \mathbb{R}P^2 \cong \mathbb{Z}/2.$$

Further,

$$\pi_k \mathbb{R}P^\infty \cong \pi_k (\mathbb{R}P^\infty)^{k+1} \cong \pi_k \mathbb{R}P^{k+1} \cong \pi_k S^{k+1} = 0.$$

## 2. THE WHITEHEAD THEOREM

The following proposition is called the *homotopy extension lifting property*. It is technical to state but will have important consequences.

**Proposition 2.1 (HELP).** *Suppose that  $(X, A)$  is a relative CW-complex of dimension  $\leq n$ . Suppose that  $e : Y \rightarrow Z$  is an  $n$ -equivalence. Given a diagram*

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow e \\ X & \longrightarrow & Z \end{array}$$

*which commutes up to a homotopy  $H$ , there exists a lift  $X \rightarrow Y$  which makes the upper triangle commute and makes the lower triangle commute up to a homotopy  $\tilde{H}$  that extends  $H$ .*

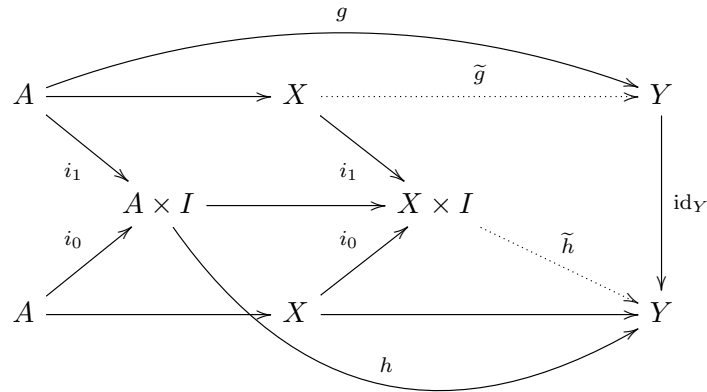
*In other words, in*

$$\begin{array}{ccccc} & & & g & \\ & & & \curvearrowright & \\ A & \longrightarrow & X & \xrightarrow{\tilde{g}} & Y \\ & \searrow & \searrow & & \downarrow e \\ & i_1 & i_1 & & \\ & A \times I & \longrightarrow & X \times I & \\ & \nearrow & \nearrow & \searrow \tilde{h} & \\ A & \xrightarrow{i_0} & X & \xrightarrow{f} & Z \\ & \searrow & \searrow & \curvearrowleft & \\ & & & h & \end{array}$$

*the dashed arrows exist.*

**Corollary 2.2.** *If  $A \rightarrow X$  is a relative CW-complex, then it is a cofibration.*

*Proof.* In

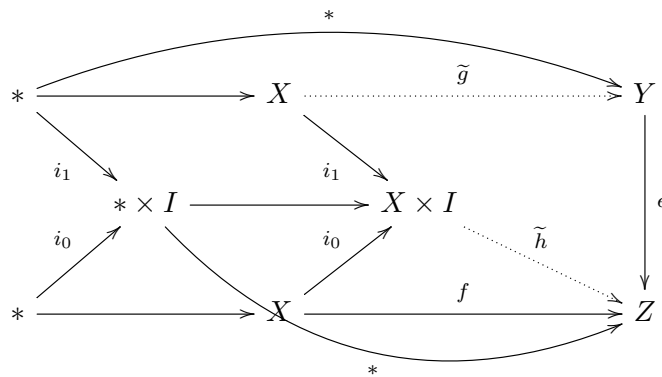


the map  $\tilde{h}$  is the desired extension. □

- Theorem 2.3** (Whitehead). (a) Let  $e : Y \rightarrow Z$  be an  $n$ -equivalence. Let  $X$  be a CW-complex of dimension  $d$ . Then  $[X, Y]_* \rightarrow [X, Z]_*$  is a bijection if  $d < n$  and a surjection if  $d = n$ .
- (b) Let  $e : Y \rightarrow Z$  be a weak equivalence and  $X$  be any CW-complex. Then  $[X, Y]_* \rightarrow [X, Z]_*$  is a bijection.
- (c) Let  $e : Y \rightarrow Z$  be an  $n$ -equivalence. Suppose that  $Y$  and  $Z$  are CW complexes and  $\dim Y, \dim Z < n$ . Then  $e$  is a homotopy equivalence.
- (d) Let  $e : Y \rightarrow Z$  be a weak equivalence of CW-complexes. Then  $e$  is a homotopy equivalence.

*Proof.*

(a) Let  $[f] \in [X, Z]_*$ . Then the diagram



gives a map  $\tilde{g} : X \rightarrow Y$  and a homotopy  $\tilde{h}$  between  $e \circ \tilde{g}$  and  $f$ . So  $e_*([g]) = [e \circ g] = [f]$  and  $e_*$  is surjective.

To be continued... □