MATH 6280 - CLASS 16

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These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. CW Complexes

Definition 1.1 (CW-complex)**.** (a) Let X_0 be a discrete set of points. Assume that X^{n-1} has been constructed. Let $\{D_i^n\}_{i\in I_n}$ be a set of n-disks D_i^n with boundary S_i^{n-1}. Let

$$\{\phi_i^n : S_i^{n-1} \to X^{n-1}\}_{i\in I_n}$$

be corresponding continuous maps which we call the *attaching maps* or *characteristic maps*. Then X_n is defined as the pushout:

$$\begin{tikzcd} \cup_{i\in I_n} S_i^{n-1} \arrow[r, "\cup\phi_i^n"] \arrow[d] & X^{n-1} \arrow[d] \\ \cup_{i\in I_n} D_i^n \arrow[r] & X^n \end{tikzcd}$$

- $X^{n-1} \subset X^n$ is a closed subspace and $X = \cup_{n=0}^{\infty} X^n$ with the union topology.
- X^n is called the n–skeleton.

We call X a CW–complex.

(b) More generally, we take $X^0 = A \cup P$ for any topological space A and discrete set of points P and build X by attaching n–disks to A inductively. Then (X, A) is a relative CW–complex.

(c) A continuous map $f : X \to Y$ between CW–complexes X and Y is called *cellular* if $f(X^n) \subset Y^n$.

(d) X has dimension $\leq n$ if $X = X^n$.

Example 1.2.

- $S^n = D^n \cup_{S^{n-1}} *$ is a CW-complex:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

- $\mathbb{R}P^n$ is a CW–complex with on cell in each dimension $0 \leq 1 \leq n$. It is obtained inductively from $\mathbb{R}P^{n-1}$ by attaching a disk D^n to $\mathbb{R}P^{n-1}$ via the double cover map $\phi : S^{n-1} \to \mathbb{R}P^{n-1}$:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\phi"] \arrow[d] & \mathbb{R}P^{n-1} \arrow[d] \\ D^n \arrow[r] & \mathbb{R}P^n \end{tikzcd}$$

- $\mathbb{C}P^n$ is obtained inductively from $\mathbb{C}P^{n-1}$ by attaching a $2n$–cell

$$\begin{tikzcd} S^{2n-1} \arrow[r, "\phi"] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d] \\ D^{2n} \arrow[r] & \mathbb{C}P^n \end{tikzcd}$$

  via the standard covering map

$$S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1}$$

  which comes from viewing S^{2n-1} as the unit sphere in $\mathbb{C}^n$. The orbit of a unit vector $(c_1, \ldots, c_n)$ under the action of $S^1 \cong \mathbb{C}^\times$ determines a line in $\mathbb{C}^n$, hence a point of $\mathbb{C}P^{n-1}$.

Remark 1.3. Here are some facts about CW–complexes that we will not prove but will use if need. Let X be a CW-complex.

- X is locally path connected
- X is Hausdorff (in fact, it is T_1 and normal).
- If K is compact and $K \subset X$, then $K \subset Y$ for a sub-complex $Y \subset X$ which has finitely many cells. In particular, $K \subset X^n$ for some n.

Remark 1.4. We have

$$X = \bigcup_n X_n = \operatorname{colim}_n X_n.$$

This implies that

$$\mathrm{Map}_*(X, Y) = \mathrm{Map}_*(\operatorname{colim}_n X_n, Y) = \lim_n \mathrm{Map}_*(X_n, Y).$$

Passing to π_0, we obtain

$$[X,Y]_* = \lim_n [X_n, Y]_*$$

2. n–equivalences

Definition 2.1.
- A space X is n–connected if $\pi_k(X,x) = 0$ for $0 \le k \le n$ and all $x \in X$.
- A pair (X,A) is n–connected if $\pi_k(X,A,a) = 0$ for $1 \le k \le n$ and $\pi_0 A \to \pi_0 X$ is surjective.
- A map $f : X \to Y$ is an n–equivalence if, for all $x \in X$,

  $$\pi_k(f) : \pi_k(X,x) \to \pi_k(Y, f(x))$$

  is an isomorphism for $0 \le k \le n-1$ and a surjection on π_n. It is a weak equivalence if it is an isomorphism for all n. Note that using the long exact sequence on homotopy groups

$$\ldots \to \pi_2(A,a) \to \pi_2(X,a) \to \pi_2(X,A) \to \pi_1(A,a) \to \pi_1(X,a) \to \pi_1(X,A) \to \pi_0(A,a) \to \pi_0(X,a)$$

  we get that (X,A) is n–connected if and only if $A \to X$ is an n–equivalence.

We will use the following exercise:

Exercise 2.2. An element $[f] \in \pi_q(X,A)$ is trivial (i.e. homotopic to the constant map at $*$) if and only if it has a representative $f : (D^q, S^{q-1}) \to (X,A)$ such that $f(D^q) \subset A$.

Example 2.3. The inclusion $S^{n-1} \to D^n$ is an n-equivalence. Indeed, $\pi_k S^{n-1} = \pi_{k-1} D^n = 0$ and $\mathbb{Z} \cong \pi_{n-1} S^{n-1} \to \pi_{n-1} D^n = 0$ is surjective.

Proposition 2.4. *If $X \cup_\phi D^{n+1}$ for some map $\phi : S^n \to X$, then $(X \cup_\phi D^{n+1}, X)$ is n–connected.*

Proof. For a careful proof, see Proposition 5.1.24.

Let $q \le n$. The idea is as follows. Let $f : (D^q, S^{q-1}) \to (X \cup_\phi D^{n+1}, X)$ represent an element in $\pi_q(X \cup_\phi D^{n+1}, X)$. Then we can deform f relative to X so that

$$(D^q, S^{q-1}) \xrightarrow{f} (X \cup_\phi D^{n+1}, X)$$

misses a point of the interior of D^{n+1}. We can assume that this point is the center of D^{n+1}. Using a retraction of $D^{n+1} - \{0\}$ onto its boundary S^n, we can deform f so that it's image lies entirely in X. Therefore, it represents the trivial element in $\pi_q(X \cup_\phi D^{n+1}, X)$. $\square$