MATH 6280 - CLASS 14

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Fibrations Continued

Proposition 1.1. *Let $p : E \to B$ be a fibration. Let F be the fiber, i.e., the pull-back*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r] & B \end{tikzcd}$$

Then the natural map $\phi : F \to P_p$

$$\begin{tikzcd} & P_p \arrow[d] \\ F \arrow[ur, dotted, "\phi"] \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r] & B \end{tikzcd}$$

given by

$$\phi(e) = (e, *) \in \{(e, \alpha) \mid \alpha(1) = p(e),\ \alpha(0) = *\} \subset E \times PB$$

is a homotopy equivalence.

Proof. First, we construct a map r

$$F \xrightarrow{\phi} P_p \xrightarrow{r} F$$

which will be the homotopy inverse of ϕ. Recall that

$$E_p = \{(e, \alpha) \mid \alpha(1) = p(e)\} \subseteq E \times B^I$$

and that

$$P_p = \{(e, \alpha) \mid \alpha(1) = p(e), \alpha(0) = *\} \subseteq E \times B^I.$$

Consider the commutative diagram

$$\begin{tikzcd} E_p \arrow[r, "\pi_E"] \arrow[d, "i_1"'] & E \arrow[d, "p"] \\ E_p \times I \arrow[r, "ev_t"'] \arrow[ur, dotted, "H"] & B \end{tikzcd}$$

The map H exists since $E \to B$ is a fibration and, letting $H_t(e, \alpha) = H((e, \alpha), t)$,

$$p(H_t(e, \alpha)) = \alpha(t).$$

Let $r : P_p \to F$ be

$$r(e, \alpha) = H_0(e, \alpha).$$

Since $(e, \alpha) \in P_p$, we have $p(H_0(e, \alpha)) = \alpha(0) = *$ so this is well-defined. We need homotopies $r \circ \phi \simeq \mathrm{id}_F$ and $\phi \circ r \simeq \mathrm{id}_{P_p}$.

Let

$$K : F \times I \to F$$

be given by

$$K_t(f) = H_t(f, *).$$

Then

$$K_0(f) = H_0(f, *) = (r \circ \phi)(f)$$

so $K_0 = (r \circ \phi)$. Further,

$$K_1(f) = H_1(f, *) = f$$

so $K_1 = \mathrm{id}_F$. Hence, K is a homotopy $r \circ \phi \simeq \mathrm{id}_F$.

Let $G : P_p \times I \to P_p$ be defined by

$$G_t(e, \alpha(s)) = (H_t(e, \alpha), \alpha(ts)).$$

Then,

$$G_1(e, \alpha(s)) = (H_1(e, \alpha), \alpha(s)) = (e, \alpha(s))$$

so $G_1 = \mathrm{id}_{P_p}$. Further,

$$G_0(e, \alpha(s)) = (H_0(e, \alpha), \alpha(0)) = (H_0(e, \alpha), *) = (\phi \circ r)(e, \alpha)$$

so $G_0 = \phi \circ r$ and G is a homotopy $\phi \circ r \simeq \mathrm{id}_{P_p}$. □

Example 1.2. $B \times F \xrightarrow{\pi_B} B$ is a fibration

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & B \times F \arrow[d, "\pi_B"] \\ A \times I \arrow[r, "H"'] & B \end{tikzcd}$$

Let

$$\widetilde{H}(a, t) = (H(a, t), \pi_F f(a)).$$

Then, $\widetilde{H}(a, 0) = f(a)$ and $\pi_B(\widetilde{H}(a, t)) = H(a, t)$ and the diagram commutes.

Definition 1.3. If there is a homotopy equivalence $\phi : E \to B \times F$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "p"'] & & B \times F \arrow[dl, "\pi_B"] \\ & B & \end{tikzcd}$$

then p is called a *homotopically trivial fibration.*

Exercise 1.4. If $E \to B$ is a fibration and B is contractible, then p is homotopically trivial.

Example 1.5. Let $p : E \to B$ be a covering space. Then p is a fibration with a unique lift $\widetilde{H}$.

To show this, we use the compactness of I, cover $A \times I$ by open neighborhoods $N_a \times I$ of $\{a\} \times I$ such that there is a partition $\{t_{a,0}, \dots, t_{a,n_a}\}$ with $H([t_{a,i}, t_{a,i+1}]) \subset U_{a,i}$ where $p^{-1}(U_i)$ is a disjoint union of open sets homeomorphic to U_i. Let $V \subset p^{-1}U_0$ be the component that contains $f(a)$. Then we can lift $H : N_a \times [0, t_{a,1}] \to B$ to E by letting $\widetilde{H}_a = p^{-1} \circ H$. By induction, we get a lift $\widetilde{H}_a : N_a \times I \to E$.

By the uniqueness of path lifting, for any $y \in N_a$, the lift $\widetilde{H} : \{y\} \times I \to E$ is unique. So, the lifts $\widetilde{H}_a$ and $\widetilde{H}_{a'}$ agree on the intersections $N_a \cap N_{a'}$, and hence we can glue them together to obtain the lift $\widetilde{H}$.

2. Fiber Bundles

Definition 2.1. Let $p : E \to B$ be a continuous surjective map. Then p is a *fiber bundle* with fiber F (also called a *locally trivial fibration*) if

(a) $p^{-1}(b) = F$ for every $b \in B$

(b) For every $b \in B$, there exists an open neighborhood $b \in U_b \subset B$ and a homeomorphism $\phi_{U_b} : p^{-1}(U_b) \to F \times U_b$ such that the following diagram commutes

$$\begin{tikzcd} p^{-1}(U_b) \arrow[rr, "\phi_{U_b}"] \arrow[dr, "p"'] & & F \times U_B \arrow[dl, "\pi_{U_b}"] \\ & U_b & \end{tikzcd}$$

Proposition 2.2. *A fiber bundle $p : E \to B$ is a Serre fibration.*

This follows from the more general claim:

Proposition 2.3. *Let $p : E \to B$. Suppose that there is an open cover $\mathcal{U}$ of B such that, for each $U \in \mathcal{U}$, the pull-back*

$$\begin{tikzcd} E_U \arrow[r] \arrow[d, "p_U"'] & E \arrow[d, "p"] \\ U \arrow[r] & B \end{tikzcd}$$

is a Serre fibration. Then p is a Serre fibration.

Proof. Consider

$$\begin{tikzcd} I^q \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ I^q \times I \arrow[r, "H"'] & B \end{tikzcd}$$

Since $I^q \times I$ is compact, so is $H(I^q \times I)$, so we can choose a finite cover $U_1, \ldots, U_k$ of $H(I^q \times I)$ by sets $U \in \mathcal{U}$. Then $\{H^{-1}(U)\}_{U \in \mathcal{U}}$ is an open cover of $I^q \times I$. Using the fact that $I^q \times I$ is a compact metric space and the Lesbesgue number lemma, we can divide I^q into cubes $\{c_q\}_{\mathcal{C}}$ and I into intervals $0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$ such that for all $c \in \mathcal{C}$ and $0 \leq i \leq m-1$ $c \times [t_i, t_{i+1}] \subset H^{-1}(U_j)$ for some $1 \leq j \leq k$.

Let $\mathcal{C}_n$ be the set of all n-facces in $\mathcal{C}$. For $c_0 \in \mathcal{C}$, $(c_0, t_0) \in U_i$ for some i. Since $p_{U_i} : E_{U_i} \to U_i$ is a fibration, we can lift H to a map $\widetilde{H} : c_0 \times [t_0, t_1] \to E_{U_i}$. Hence, this gives a lift

$$\widetilde{H} : \cup_{\mathcal{C}_0} c_0 \times [t_0, t_1] \to E.$$

Now suppose that we have lifted

$$\widetilde{H}_{n-1} : \cup_{\mathcal{C}_{n-1}} c_{n-1} \times [t_0, t_1] \to E.$$

Let $c_n \in \mathcal{C}_n$ and $c_n \times [t_0, t_1] \subset U_i$. Then f together with the lift on $\mathcal{C}_{n-1} \times [t_0, t_1]$ specify

$$\widetilde{H}'_{n-1} : c_n \times \{0\} \cup \partial c_n \times [t_0, t_1] \to E.$$

Choose a homeomorphism

$$\phi : c_n \times \{0\} \cup \partial c_n \times [t_0, t_1] \to c_n \times \{0\}$$

Then, using the fact that $p_{U_i} : E_{U_i} \to U_i$, we get a lift:

$$\begin{tikzcd}
c_n \times \{0\} \cup \partial c_n \times [t_0, t_1] \arrow[r, "\widetilde{H}'_{n-1}"] \arrow[d, "i_0 \circ \phi"'] & E_{U_i} \arrow[d, "p_{U_i}"] \\
c_n \times [t_0, t_1] \arrow[r] \arrow[ur, dotted, "\widetilde{H}_n"] & U_i
\end{tikzcd}$$

Since $c_n \times [t_0, t_1]$ and $c'_n \times [t_0, t_1]$ can only intersect on

$$(c_n \times \{0\} \cup \partial c_n \times [t_0, t_1]) \cap (c'_n \times \{0\} \cup \partial c'_n \times [t_0, t_1])$$

where the lift was already consistently specified, this gives us well-defined lift:

$$\widetilde{H}_n : \cup_{\mathcal{C}_n} c_n \times [0, t_1] \to E.$$

Inductively, we get a lift

$$\widetilde{H}_q : \cup_{\mathcal{C}_q} c_q = I^q \times [0, t_1] \to E.$$

Now, proceed by induction to lift H to $I^q \times [0, t_j]$. □

Definition 2.4. An open cover of B is locally finite if every point $b \in B$ has an open neighborhood that intersects finitely many elements of the cover. A space B is paracompact if every open cover has a locally finite open refinement that covers B.

Theorem 2.5 (Hurewicz - *On the concept of fiber space*, Proc. Nat. Acad. Sci. USA 41 (1955) 956–961)**.** *Let $p : E \to B$ be a local Hurewicz fibration. If B is a paracompact topological space, then p is a Hurewicz fibration.*

Example 2.6. If G is a Lie group with closed subgroup H, then $G \xrightarrow{p} G/H$ is a fiber bundle. The fiber at each point is H.

3. Long exact sequence on homotopy groups for a pair

Definition 3.1. Let $A \xrightarrow{i} X$ be the inclusion of a subspace which contains the base point. Let

$$P_i = \{(a, \alpha) \mid \alpha(0) = *,\ \alpha(1) = a\} = \{\alpha \in PX \mid \alpha(1) \in A\}$$

be its homotopy fiber. For $n \geq 1$, let

$$\pi_n(X, A) = \pi_n(X, A, *) = \pi_{n-1} P_i.$$

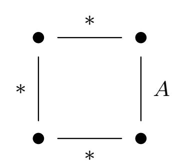
Remark 3.2. $\pi_n(X, A)$ can also be viewed as homotopy classes of maps of pairs $(I^n, \partial I^n, J^n)$ for $J^n = \partial I^{n-1} \times I \cup I^{n-1} \times \{0\}$. Here, if I^n is a box, ∂I^n is its boundary and J^n is the boundary with the interior of a face deleted.

Here are some examples:

- $\pi_0 P_i$ are homotopy classes of maps which start at the base point $*$ and end at some point in a
- $\pi_1 P_i = \pi_0 \Omega P_i$. Now, a loop in P_i is a path of paths, so can be viewed as a map

  $$\phi : I^2 \to X$$

  On $\phi(0, t) = *$ and $\phi(1, t) = *$ since the constant path is the base point of P_i. For $0 < s < 1$, $\phi(s, 0) = *$ and $\phi(s, 1) \in A$ since $\phi(s, t) \in P_i$. So, in other words, on J^2, the map is the base point and on ∂I^2 the image is a subset of A.



We can also $\pi_n(X, A)$ as homotopy classes of maps of pairs $(D^n, S^{n-1}, *) \to (X, A, *)$.