MATH 6280 - CLASS 13

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Fibrations

Definition 1.1. A map $p : E \to B$ has the*homotopy lifting property with respect to $\mathcal{C}$* if, for every A in $\mathcal{C}$ and every diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & X
\end{tikzcd}$$

there is a map $\widetilde{H}$ making the diagram commute. Equivalently, given the diagram

$$\begin{tikzcd}
A \arrow[dr, dotted, "\widetilde{h}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "h"'] & & \\
& E^I \arrow[r, "ev_0"] \arrow[d, "p_*"] & E \arrow[d, "p"] \\
& B^I \arrow[r, "ev_0"'] & B
\end{tikzcd}$$

there is a map $\widetilde{h} : A \to E^I$ making the diagram commute.

2. Homotopy theoretic examples

Proposition 2.1. *If $p : E \to B$ is a fibration and $f : X \to B$ is a continuous map, then the pull-back $X \times_B E \to X$ is a fibration*

$$\begin{tikzcd}
X \times_B E \arrow[r] \arrow[d] & E \arrow[d] \\
X \arrow[r, "f"'] & B
\end{tikzcd}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
Z \arrow[drr, dotted, "\star"] \arrow[drrrr, bend left] \arrow[ddr, dotted, "*"'] \arrow[dddrr, bend left] & & & & \\
& & (X \times_B E)^I \arrow[rr] \arrow[dl] \arrow[dd] & & X \times_B E \arrow[dl] \arrow[dd] \\
& E^I \arrow[rr] \arrow[dd] & & E \arrow[dd] & \\
& & X^I \arrow[rr] \arrow[dl] & & E \arrow[dl] \\
& B^I \arrow[rr] & & B &
\end{tikzcd}$$

We get $*$ from the fact that $p : E \to B$ is a fibration and $\star$ from the fact that

$$\begin{tikzcd}
(X \times_B E)^I \arrow[r] \arrow[d] & E^I \arrow[d] \\
X^I \arrow[r] & B^I
\end{tikzcd}$$

is a pull-back. □

Exercise 2.2. Check that if $E = X \times_Z Y$ then $E^I = X^I \times_{Z^I} Y^I$.

Proposition 2.3. *Let*

$$E_{f,0} = \{(x, \alpha) \in X \times Y^I \mid f(x) = \alpha(0)\}$$

be the pull-back

$$\begin{tikzcd}
E_{f,0} \arrow[r] \arrow[d] & Y^I \arrow[d, "ev_0"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Then the map $E_{f,0} \xrightarrow{ev_1} Y$ is a fibration.

Proof. Consider a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & E_{f,0} \arrow[d, "ev_1"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & Y \end{tikzcd}$$

Let $f(a) = (x_a, \alpha_a)$. Then $H(a,0) = \alpha_a(1)$. Let

$$\beta_a(t) = H(a,t) \in Y^I.$$

Then, $\beta_a(0) = \alpha_a(1)$. Define

$$\gamma_{a,s} = \begin{cases} \alpha_a(t(1+s)) & t \leq \frac{1}{1+s} \\ \beta_a(t(1+s)-1) & t \geq \frac{1}{1+s}. \end{cases}$$

Let

$$\widetilde{H}(a,s) = (x_a, \gamma_{a,s}).$$

Then,

$$ev_1(\widetilde{H}(a,s)) = \gamma_{a,s}(1) = \beta_a(s) = H(a,s)$$

and the diagram commutes. □

Exercise 2.4. Suppose that $p : E \to B$ is a fibration and that $f : E' \to E$ is a homeomorphism. Check that $p \circ f : E' \to B$ is a fibration.

Example 2.5. The following are fibrations:

- $Y^I \xrightarrow{ev_1} Y$ since $Y^I \cong E_{\mathrm{id}_Y,0}$
- $E_f \xrightarrow{ev_0} Y$ is a fibration since it is the composite $E_f \xrightarrow{\tau} E_{f,0} \xrightarrow{ev_1} Y$ where

  $$\tau(x,\gamma) = (x, \tau(\gamma))$$

  where $\tau(\gamma(t)) = \gamma(1-t)$ and τ is a homeomorphism.
- $Y^I \xrightarrow{ev_0} Y$ since it is E_{id_Y}.
- $PY \xrightarrow{ev_1} Y$ is a fibration since it is E_f for $f : * \to Y$:

$$\begin{tikzcd} PY \arrow[r] \arrow[d] & Y^I \arrow[d, "ev_0"] \\ * \arrow[r] & Y \end{tikzcd}$$

- $P_f \to X$ since it is the pull–back

$$\begin{tikzcd} P_f \arrow[r] \arrow[d, "p"'] & PY \arrow[d, "ev_1"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

- $E_f \xrightarrow{p} X$ since it is the pull-back

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, "p"] & Y^I \arrow[d, "ev_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Next time, we will prove the following result.

Proposition 2.6. *Let $p : E \to B$ be a fibration. Let F be the fiber, i.e., the pull-back*

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ * \arrow[r] & B \end{tikzcd}$$

Then the natural map $\phi : F \to P_p$

$$\begin{tikzcd} & P_p \arrow[d] \\ F \arrow[r] \arrow[d] \arrow[ur, dotted, "\phi"] & E \arrow[d] \\ * \arrow[r] & B \end{tikzcd}$$

given by

$$\phi(e) = (e, *) \in \{(e, \alpha) \mid \alpha(1) = p(e),\ \alpha(0) = *\} \subset E \times PB$$

is a homotopy equivalence.

We have the following consequences.

Corollary 2.7. *Consider $P_p \to P_f \xrightarrow{p} X \xrightarrow{f} Y$. The natural map $\Omega Y \to P_p$ is a homotopy equivalence.*

Proof. Indeed, ΩY is the fiber of the fibration $P_f \xrightarrow{p} X$. □

Remark 2.8. Suppose that $E \xrightarrow{p} B$ is a fibration. Let

$$P_{p,b} = \{(e, \alpha) \mid \alpha(1) = p(e), \alpha(0) = b\} \subset E \times B^I$$

be the homotopy fiber over $b \in B$ and $F_b = p^{-1}(b)$ be the fiber over b. Then

$$F_b \simeq P_{p,b}$$

by the previous results.

If b_1 and b_2 are in the same path component of B, then it's simple to check that $P_{p,b_1} \simeq P_{p,b_2}$. This implies that for a fibration, the fibers over each point are homotopy equivalent.