# MATH 6280 - CLASS 12

Contents



These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

## 1. Homotopy Fiber

**Construction.**
- *The* mapping space $Y^I = \mathrm{Map}(I, Y)$ *with maps* $ev_t : Y^I \to Y$, $ev_t(\alpha) = \alpha(t)$. *If* $Y$ *is based, the constant map at* $*$ *is a base point for* $Y^I$.
- *Given a point* $* \to Y$, *the* path space

$$\begin{array}{ccc} PY & \longrightarrow & Y^I \\ \downarrow & & \downarrow{\scriptstyle ev_0} \\ * & \longrightarrow & Y \end{array}$$

*There is a map* $PY \xrightarrow{ev_1} Y$ *and the constant map is a natural base point.*
- *The* loop space

$$\begin{array}{ccc} \Omega Y & \longrightarrow & PY \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ * & \longrightarrow & Y \end{array}$$

- *The* mapping path space $E_f = \{(x, \alpha) \mid f(x) = \alpha(1)\}$

$$\begin{array}{ccc} E_f & \longrightarrow & Y^I \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ X & \xrightarrow[f]{} & Y \end{array}$$

  *with base point* $(*, *)$ *with map* $E_f \xrightarrow{ev_0} Y$*. Note, we can also form a homeomorphic space* $E_{f,0} = \{(x, \alpha) \mid f(x) = \alpha(0)\}$ *with a map* $E_{f,0} \xrightarrow{ev_1} Y$*.*
- *For a based map* $f : X \to Y$*, the fiber of over* $* \in Y$ *is* $F = f^{-1}(*)$*, that is, the pull-back*

$$\begin{array}{ccc} F & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

- *The* homotopy fiber $P_f = \{(x, \alpha) \mid f(x) = \alpha(1),\ \alpha(0) = *\}$

$$\begin{array}{ccc} P_f & \longrightarrow & PY \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle ev_1} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \text{or} \qquad \begin{array}{ccc} P_f & \longrightarrow & E_f \\ \downarrow & & \downarrow{\scriptstyle ev_0} \\ * & \longrightarrow & Y \end{array}$$

  *(also often denoted* $Ff$*,* $F_f$*, ...). The map* $p : P_f \to X$ *sends* $(x, \alpha)$ *to* $x$*.*

**Lemma 1.1.** *A map* $f : X \to Y$ *is homotopic to the constant map at a point* $* \to Y$ *if and only if it lifts to* $PY$*, that is,*

$$\begin{array}{ccc} & & PY \\ & \nearrow & \downarrow{\scriptstyle ev_1} \\ X & \xrightarrow[f]{} & Y \end{array}$$

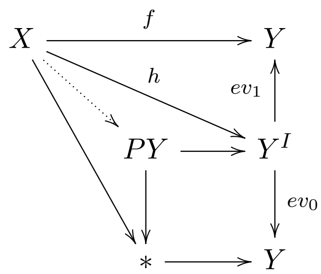
*(If* $f$ *is a map of based spaces and it is null via a based point preserving homotopy if and only if there is a based lift to* $PY$*.)*

*Proof.* A null-homotopy is equivalent to a map $h : X \to Y^I$ such that

$$h(x)(1) = f(x)$$

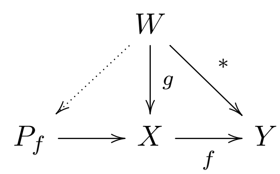
$$h(x)(0) = *.$$

Consider the following diagram. Since the square is a pull-back,

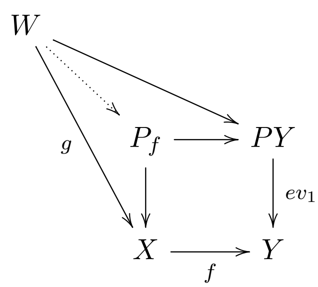



there must be an extension of $f$ to $PY$. □

**Proposition 1.2.** *Consider maps $W \xrightarrow{g} X \xrightarrow{f} Y$. Then $f \circ g$ is null-homotopic if and only if $g$ extends to the homotopy fiber of $f$, that is*




*Proof.* The following diagram shows that an extension of $g$ to $P_f$ is equivalent to an extension of $f \circ g$ to to $PX$, which is equivalent to $f \circ g$ being null-homotopic.

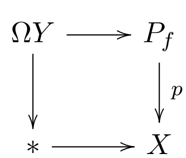



□

**Remark 1.3.** There is an inclusion $\Omega Y \hookrightarrow P_f$:

$$\alpha \mapsto (*, \alpha).$$

In fact, there is a pull-back



Further, $\Omega Y \hookrightarrow P_f$ lifts to $P_p$

$$\begin{array}{ccc} & & P_p \\ & \nearrow & \downarrow \\ \Omega Y & \longrightarrow & P_f \end{array}$$

where

$$\alpha \mapsto ((*, \alpha), *) \in P_p \subset P_f \times PX.$$

We will show that

- $\Omega Y \to P_p$ is a homotopy equivalence.
- For $P_{p_2} \xrightarrow{p_3} P_{p_1} \xrightarrow{p_2} P_f \xrightarrow{p_1} X \xrightarrow{f} Y$, the following commutes up to homotopy

$$\begin{array}{ccc} \Omega X & \xrightarrow{-\Omega f} & \Omega Y \\ \downarrow & & \downarrow \\ P_{p_2} & \xrightarrow[p_3]{} & P_{p_1} \end{array}$$

Dual to the Barratt–Puppe sequence, we thus get a commutative diagram

$$\begin{array}{ccccccccccccccc} \dots & \longrightarrow & \Omega^2 Y & \xrightarrow{-\Omega i} & \Omega P_f & \xrightarrow{-\Omega q_1} & \Omega X & \xrightarrow{-\Omega f} & \Omega Y & & & & & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \overset{i}{\searrow} & & & & & \\ \dots & \longrightarrow & P_{p_4} & \xrightarrow{p_5} & P_{p_3} & \xrightarrow{p_4} & P_{p_2} & \xrightarrow{p_3} & P_{p_1} & \xrightarrow{p_2} & P_f & \xrightarrow{p_1} & X & \xrightarrow{f} & Y \end{array}$$

**Proposition 1.4.** *For $f : X \to Y$ a map of pointed spaces, there is a long exact sequence*

$$\dots \longrightarrow [Z, \Omega P f]_* \longrightarrow [Z, \Omega X]_* \longrightarrow [Z, \Omega Y]_* \longrightarrow [Z, P_f]_* \longrightarrow [Z, X]_* \longrightarrow [Z, Y]_*$$

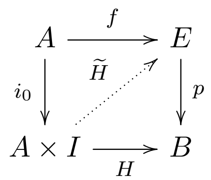
**Corollary 1.5.** *There is a long exact sequence on homotopy groups*

$$\dots \longrightarrow \pi_1 P_f \longrightarrow \pi_1 X \longrightarrow \pi_1 Y \longrightarrow \pi_0 P_f \longrightarrow \pi_0 X \longrightarrow \pi_0 Y$$

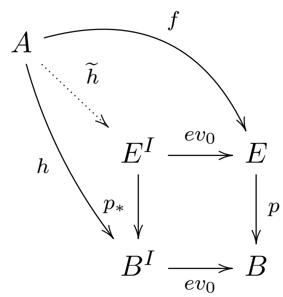
**Remark 1.6.** We will also show that if $X \to Y$ is a special kind of map called a fibration, then $F \to P_f$ is a homotopy equivalence, so that, for fibrations, we get a long exact sequence as above with $P_f$ replaced by $F$.

## 2. Fibrations

**Definition 2.1.** A map $p : E \to B$ has the*homotopy lifting property with respect to* $\mathcal{C}$ if, for every $A$ in $\mathcal{C}$ and every diagram




there is a map $\widetilde{H}$ making the diagram commute. Equivalently, given the diagram



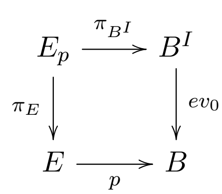

there is a map $h : A \to E^I$ making the diagram commute.

**Definition 2.2.** If $\mathcal{C} = \text{Top}$, a map $p : E \to B$ which has the HLP with respect to $\mathcal{C}$ is called a fibration or *Hurewicz fibration.* If $\mathcal{C} = \{I^n\} = \{D^n\}$, or equivalently, CWTop, then $p$ is a called a *Serre fibration.*

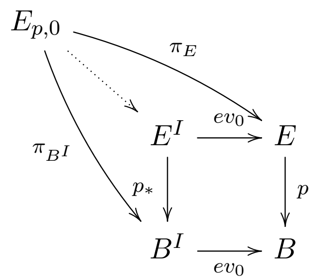
**Proposition 2.3.** *Let*

$$E_{p,0} = \{(e, \gamma) \mid \gamma(0) = p(e)\} \subset E \times B^I,$$

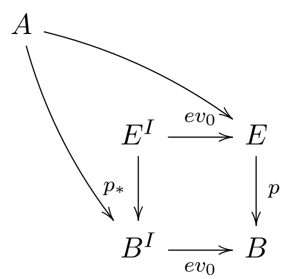
*i.e., the pull-back*



*A map $E \xrightarrow{p} B$ is a fibration if and only if there is a map $s : E_{p,0} \to E^I$ making the following diagram commute:*




*Proof.* If $p$ is a fibration, this is the HLP. If there is such a map, then, given a diagram




the universal property of the pull-back gives a map $Y \xrightarrow{f} E_{p,0}$. Then $\widetilde{p} = s \circ f$ has the desired properties. $\square$