MATH 6280 - CLASS 11

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Barratt-Puppe sequence

We now have a diagram which commutes up to homotopy

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r, "i_1"] & C_f \arrow[r, "i_2"] \arrow[dr, "\pi"'] & C_{i_1} \arrow[r, "i_3"] \arrow[d, "q_1"'] & C_{i_2} \arrow[r, "i_4"] \arrow[d, "q_2"'] & C_{i_3} \arrow[r, "i_5"] \arrow[d, "q_3"'] & C_{i_4} \arrow[r, "i_6"] \arrow[d, "q_4"'] & C_{i_5} \arrow[r] \arrow[d, "q_5"'] & \dots \\
 & & & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \arrow[r, "-\Sigma i_1"] & \Sigma C_f \arrow[r, "-\Sigma \pi"] & \Sigma^2 X \arrow[r, "\Sigma^2 f"] & \Sigma^2 Y \arrow[r] & \dots
\end{tikzcd}$$

and in which every three term sequence is a cofiber sequence up to homotopy equivalence.

Exercise 1.1. There are homeomorphism $\Sigma C_f \cong C_{\Sigma f} \cong C_{-\Sigma f}$.

Theorem 1.2. *Let Z be a based topological space. There is a long exact sequence*

$$\dots \longrightarrow [\Sigma C_f, Z]_* \longrightarrow [\Sigma Y, Z]_* \longrightarrow [\Sigma X, Z]_* \longrightarrow [C_f, Z]_* \longrightarrow [Y, Z]_* \longrightarrow [X, Z]_*$$

Proof. Recall that $A \xrightarrow{p} B \xrightarrow{q} C$ is null if and only if the following extension problem has a solution

$$\begin{tikzcd}
A \arrow[r, "p"] & B \arrow[r, "i"] \arrow[d, "g"'] & C_p \arrow[dl, dotted] \\
 & C &
\end{tikzcd}$$

This is equivalent to saying that

$$[C_p, C] \xrightarrow{i^*} [B, C] \xrightarrow{p^*} [A, C]$$

is exact.

Since every three term sequence of

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma Y \longrightarrow \dots$$

is a cofiber sequence up to homotopy equivalences, the claim follows. □

Remark 1.3. If $X \to Y$ is a cofibration, then $C_f \simeq Y/X$ and we get a long exact sequence

$$\dots \longrightarrow [\Sigma(Y/X), Z]_* \longrightarrow [\Sigma Y, Z]_* \longrightarrow [\Sigma X, Z]_* \longrightarrow [Y/X, Z]_* \longrightarrow [Y, Z]_* \longrightarrow [X, Z]_*$$

Remark 1.4.
- An important example of a cofibration is $A \subset X$ where X a CW-complex and A is a subcomplex.
- Let E be a space. The functor $[-, E]_* : \mathrm{Top}_*^{op} \to \mathrm{Sets}$ turn cofiber sequences $X \to Y \to C_f$ into long exact sequences.

2. Neighborhood Deformation Retracts

Definition 2.1 (Neighborhood Deformation Retracts)**.** A pair (X, A) where A is a subspace of X is an Neighborhood Deformation Retracts (NDR)-pair if there is a homotopy $H : X \times I \to X$ and a function $\varphi : X \to I$ such that

- $A = \varphi^{-1}(0)$
- $H|_{X\times\{0\}} = \mathrm{id}_X$
- $H(a, t) = a$ for all $a \in A$, $t \in I$
- $H(x, 1) \in A$ if $\varphi(x) < 1$.

If $\varphi(x) < 1$ for all $x \in X$, then (X, A) is a Deformation Retracts (DR)-pair.

Theorem 2.2. *For a closed subspace A of X, the inclusion $i : A \to X$ is a cofibration if and only if (X, A) is an NDR pair.*

Proof. If $A \xrightarrow{i} X$ is a cofibration, then there is a retraction

$$X \times I \xrightarrow{r} X \times \{0\} \cup A \times I \subset X \times I.$$

Define,

$$\varphi(x) = \sup_{t \in I} |t - p_I(r(x, t))| \qquad H(x, t) = p_X(r(x, t)).$$

- If $\varphi(x) = 0$, then $t = p_I(r(x,t))$ for all t. Hence, $r(x,t) \in A \times I$ for all $t > 0$. But A is closed in X, so $A \times I$ is closed in $X \times I$. Hence, $r(x,0) \in A \times I$. But $r(x,0) = (x,0)$ so $x \in A$.
- $H(x,0) = p_X(r(x,0)) = p_X(x,0) = x$
- $H(a,t) = p_X(r(a,t)) = p_X(a,t) = a$
- If $H(x,1) \notin A$, then $p_X(r(x,1)) \notin A$, so $r(x,1) \in X \times \{0\}$. Therefore, $p_I(r(x,1)) = 0$ and

$$1 \geq \phi(x) = \sup_{t \in I} |t - p_I(r(x,t))| \geq |1 - p_I(r(x,1))| = 1.$$

  Hence, $\phi(x) = 1$. Therefore, if $\phi(x) < 1$, then $H(x,1) \in A$.

Now suppose that (X,A) is an NDR–pair. Let $r : X \times I \to X \times \{0\} \cup A \times I$ be given by

$$r(x,t) = \begin{cases} (H(x,1), t - \varphi(x)) & t \geq \varphi(x) \\ (H(x, t/\varphi(x)), 0) & t \leq \varphi(x). \end{cases}$$

□

3. Homotopy Fiber

Construction.
- *The* mapping space $Y^I = \mathrm{Map}(I,Y)$ *with maps* $ev_t : Y^I \to Y$, $ev_t(\alpha) = \alpha(t)$. *If* Y *is based, the constant map at* $*$ *is a base point for* Y^I.
- *Given a point* $* \to Y$, *the* path space

$$\begin{tikzcd} PY \arrow[r] \arrow[d] & Y^I \arrow[d, "ev_0"] \\ * \arrow[r] & Y \end{tikzcd}$$

  There is a map $PY \xrightarrow{ev_1} Y$ *and the constant map is a natural base point.*
- *The* loop space

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & PY \arrow[d, "ev_1"] \\ * \arrow[r] & Y \end{tikzcd}$$