MATH 6280 - CLASS 1

PERSONAL NOTES: USE WITH CARE.

Contents

1. Introduction

In this class, we will study the category of “nice” topological spaces Top with morphisms the continuous maps. All maps $X \to Y$ between topological spaces are continuous unless otherwise specified. We will also study the category of based topological spaces Top_*, which are spaces with a “nice” base point $*$ and that maps preserve this base point.

1.1. **Homotopy equivalence and weak homotopy equivalence.**

Definition 1.1 (Homotopies)**.** Let $I = [0, 1]$ be the closed unit interval.

(1) A *homotopy* between $f, g : X \to Y$ is a map $H : X \times I \to Y$ such that $f = H_0 : X \times \{0\} \to Y$ and $g = H_1 : X \times \{1\} \to Y$.
(2) Spaces X and Y are *homotopy equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We write $X \simeq Y$ if they are homotopy equivalent.
(3) We also consider the notion of based homotopies and the corresponding notion of based equivalences between objects of Top_*. A based homotopy $H : X \times I \to Y$ is one where $H_t : X \times \{t\} \to Y$ preserves the base point for all $t \in I$.

(4) We let $[X, Y] = \mathrm{Hom}_*(X, Y)/\simeq$ be the set of based continuous maps from X to Y modulo based homotopy equivalences.

Algebraic topology is the study of the category of Top or Top_* up to homotopy equivalences via homotopy invariant functors from Top to some "algebraic" category such as abelian groups.

Example 1.2. (1) The n'th homology functor $H_n : \mathrm{Top} \to \mathrm{Ab}$ or the n'th cohomology functor $H^n : \mathrm{Top}^{op} \to \mathrm{Ab}$

(2) For fixed based spaces X or Y, the functors $[X, -] : \mathrm{Top}_* \to \mathrm{Ab}$ and $[-, Y] : \mathrm{Top}_*^{op} \to \mathrm{Ab}$.

(3) The n'th homotopy group functor $\pi_n : \mathrm{Top}_* \to \mathrm{Ab}$ where $\pi_n(X) = [S^n, X]$.

(4) The K–theory functor $K : \mathrm{Top} \to Ab$.

There is a weaker notion of equivalence between two topological spaces which can be easier to track down.

Definition 1.3. A map $f : X \to Y$ is an *n–equivalence* if $\pi_q f$ is an isomorphisms for $q < n$ and surjective for $q = n$. A map f is a *weak homotopy equivalence* if $\pi_n f$ is an isomorphism for all n.

Slogan. Weak homotopy equivalences are to spaces what quasi-isomorphisms are to chain complexes.

Example 1.4. One can prove that any homotopy equivalence is a weak homotopy equivalence, but the converse is not always true. One can prove that the circle S^1 and the poset P:

$$\begin{tikzcd} & \bullet_b & \\ \bullet_c \arrow[ur] \arrow[dr] & & \bullet_d \arrow[ul] \arrow[dl] \\ & \bullet_a & \end{tikzcd}$$

(with open sets $\mathcal{U} = \{\emptyset, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$) are weakly homotopy equivalent, where P has the poset topology. However, they are not homotopy equivalent. To prove this, one can use the following results:

Proposition 1.5 (May, "Weak Equivalences and Quasi-fibrations", Cor 1.4)**.** *Let $f : X \to Y$ be a map and let $\mathcal{O}$ be an open cover of Y which is closed under finite intersections. If $f : f^{-1}(U) \to U$ is a weak equivalence for all $U \in \mathcal{O}$, then $f : X \to Y$ is a weak equivalence.*

We often work with the weaker notion as it is slightly more algebraic. However, for a very wide class of spaces, the two notion coincide.

1.2. **CW-Complex and the Whitehead theorem.**

Definition 1.6. A CW–complex is a space built by gluing disks together along their boundary. If the largest kind of disk used to build X is D^n, then $\dim X = n$.

Example 1.7. The following admit the structure of CW–complexes: S^n, $\mathbb{C}P^n$, $\mathbb{R}P^n$, differentiable manifolds, algebraic and projective varieties, anything that can be triangulated.

Theorem 1.8 (Whitehead)**.** *A weak equivalence between CW complexes is a homotopy equivalence.*

Remark 1.9. Not every two CW–complexes which have the same homotopy groups are homotopy equivalent. You really need the map. For example, you can show that $S^2 \times \mathbb{R}P^3$ and $\mathbb{R}P^2 \times S^3$ have the same homotopy groups. Compute their cohomology to check that they are not homotopy equivalent.

In fact, in many cases, the following theorem justifies restricting our attention to the much nicer category of CW–complexes CWTop.

Theorem 1.10 (CW-approximation)**.** *There is a functor* $\Gamma : \mathrm{Top} \to \mathrm{CWTop}$ *and a natural transformation* $\gamma : \Gamma \to Id$ *such that* $\gamma : \Gamma X \to X$ *is a weak homotopy equivalence for any* X*.*

1.3. **Σ-Ω adjunction.**

- Let $\mathrm{Map}(X, Y)$ be the space of maps from X to Y.
- Let $\mathrm{Map}_*(X, Y)$ be the space of based maps from X to Y.
- Let $X \times Y$ be the product of X and Y.
- Let $X \wedge Y = X \times Y/(* \times Y \cup X \times *)$.

Theorem 1.11. *For* $X, Y, Z \in \mathrm{Top}$*, there is a homeomorphism*

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

$$f((x, y)) = z \mapsto f(x)(y) = z.$$

For $X, Y, Z \in \mathrm{Top}_*$*, there is a homeomorphism*

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)).$$

In particular, this implies that, for

- Let $\Omega X = \mathrm{Map}_*(S^1, X)$.
- Let $\Sigma X = S^1 \wedge X$

we have

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

This will fit into the theme of a certain duality between "mapping in" and "mapping out".

1.4. **Cofibrations and Fibrations.** In homological algebra, if we have an exact sequence

$$0 \to A_* \to B_* \to C_* \to 0$$

of chain complexes, we obtain a long exact sequence in homology H_*. This is an extremely useful computational tool! We would like to have similar tools in homotopy theory. For that, we need to know what we mean by an *exact sequence*. This will not be a word for word analogy.

There are special classes of maps called *cofibrations* and *fibrations*. Here are two prototypical examples

- The inclusion $A \hookrightarrow X$ of a subspace which has neighborhood which is a deformation retract is a cofibration. For example, the inclusion of a sub-CW-complex is a cofibration.
- A covering space $p : E \to B$ is a fibration.

Slogan. Cofibrations are very nice inclusions and fibrations are very nice surjections.

We will give very precise definitions of these notions, but here are the results:

Theorem 1.12.
If $A \to X$ is a cofibration, then for any space Z, there is a long exact sequence

$$\ldots \to [\Sigma X/A, Z] \to [\Sigma X, Z] \to [\Sigma A, Z] \to [X/A, Z] \to [X, Z] \to [A, Z]$$

If $E \to B$ is a fibration and F is the fiber, then for any space Z, there is a long exact sequence

$$\ldots \to [Z, \Omega E] \to [Z, \Omega B] \to [Z, F] \to [Z, E] \to [Z, B]$$

We can apply this theorem when $Z = S^0$, noting that

$$[S^n, Z] \cong [S^{n-1}, \Omega Z] \cong \ldots \cong [S^1, \Omega^{n-1} Z] \cong [S^0, \Omega^n Z].$$

We get:

Corollary 1.13. *If $E \to B$ is a fibration and F is the fiber, there is a long exact sequence on homotopy groups*

$$\ldots \to \pi_1 F \to \pi_1 E \to \pi_1 B \to \pi_0 F \to \pi_0 E \to \pi_0 B$$

Remark 1.14. Given any map $X \to Y$, we can either replace it by a fibration or a cofibration.

1.5. **The Freudenthal Suspensions Theorem.** Another big result that helps us understand homotopy groups better it the Freudenthal Suspension Theorem.

Definition 1.15. A space is n–connected if $\pi_q Y = 0$ for $q \leq n$.

Theorem 1.16 (Freudenthal Suspension Theorem)**.** *Let $n \geq 2$. If Y is n–connected and X is a CW complex of dimension q, then the natural map*

$$[X, Y] \to [\Sigma X, \Sigma Y]$$

is an isomorphism for $q \leq 2n$ and a surjection for $q = 2n + 1$.

You can apply this with $Y = S^n$ and conclude

Corollary 1.17. *The natural map*

$$\pi_q X \to \pi_{q+1} \Sigma X$$

is an isomorphism for $q \leq 2n - 2$ and a surjection for $q = 2n - 1$.

Remark 1.18. This is the gateway to *stable homotopy theory.* Homotopy theorist call something stable if it is invariant under suspension Σ.

1.6. **The Hurewicz Theorem.** Despite all of these tools, the following fact is still true in general:

Slogan. Homotopy groups are hard to compute while homology is easy to compute.

This is why the following result and its corollary are extremely important. They also highlight the advantage of working with CW–complexes.

Theorem 1.19 (Hurewicz)**.** *If X is $(n-1)$–connected, with $n \geq 2$, then $\widetilde{H}_i(X) = 0$ for $i < n$ and there is an isomorphism*

$$\pi_n X \to H_n(X).$$

(If $n = 1$, we get a similar result but then $H_1(X) \cong \pi_1 X/[\pi_1 X, \pi_1 X]$)

Corollary 1.20 (Whitehead)**.** *A map $f : X \to Y$ between simply-connected CW-complexes is a homotopy equivalence if and only if $H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

1.7. **Building spaces from fibrations.** A CW–complex is a space which is built out of disk D^n. If X has dimension n and A is the $n-1$–dimensional subcomplex, then

$$X/A \cong \bigvee S^n.$$

So, CW complexes are built out of spheres through cofibrations. Spheres have the wonderful property that

$$\widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

There is also a notion dual to spheres. This is the notion of Eilenberg–MacLane spaces. Let G be a group if $n = 1$ and an abelian group if $n > 1$. Then there is a space $K(G, n)$ such that

$$\pi_k K(G, n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Dual to CW–complexes, one can build spaces out of Eilenberg–MacLane spaces through fibrations. This is the Postnikov tower of a space, and we'll see how that can be used to do some computations.