NOTES ON THE BROWN REPRESENTABILITY THEOREM

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1. Hom functors

Let $\mathscr{C}$ be the category whose objects are pointed, path-connected CW-complexes, and whose morphisms are continuous maps modulo homotopy.

There is also a category $\mathbf{Set}_*$ of **pointed sets**. Its objects are pairs, (X, x) of a set X and an element $x \in X$. Its morphisms $f : (X, x) \to (Y, y)$ are set functions $f : X \to Y$ so that $f(x) = y$.

Recall that for any object X of $\mathscr{C}$, we get a functor $\mathrm{Hom}_{\mathscr{C}}(-, X) : \mathscr{C}^{op} \to \mathbf{Set}$. It takes an object Y to $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ and a morphism $\phi : Y \to Z$ to the set function $f \mapsto f \circ \phi$ from $\mathrm{Hom}_{\mathscr{C}}(Z, X) \to \mathrm{Hom}_{\mathscr{C}}(Y, X)$.

This can be upgraded to a functor $\mathrm{Hom}_{\mathscr{C}}(-, X) : \mathscr{C}^{op} \to \mathbf{Set}_*$ by taking the distinguished element of each $\mathrm{Hom}_{\mathscr{C}}(Y, X)$ to be the map-to-basepoint map. Since precomposition takes the map-to-basepoint map to the map-to-basepoint map, the morphisms already defined preserve the distinguished point of the set, as required.

2. Properties of Hom functors

The functor $\mathrm{Hom}(-, X)$ has two special properties.

(1) (Wedge property)

$$\mathrm{Hom}(\vee_{\alpha \in A} Y_\alpha, X) \cong \prod_{\alpha \in A} \mathrm{Hom}(Y_\alpha, X).$$

This is because the wedge of spaces is the coproduct in $\mathscr{C}$.

(2) (Mayer-Vietoris property) Suppose $(Y; A, B)$ is a CW-triad. Suppose $[f] \in \mathrm{Hom}(A, X)$ and $[g] \in \mathrm{Hom}(B, X)$ are functions so that $f|_{A \cap B} \simeq g|_{A \cap B}$. Then there is $[h] \in \mathrm{Hom}(Y, X)$ so that $h|_A \simeq f$ and $h|_B \simeq g$.

Proof. Let $H : A \cap B \times I \to X$ be a homotopy from $f|_{A \cap B}$ to $g|_{A \cap B}$. Then since $A \cap B \to A$ is a cofibration, H extends to a homotopy $\tilde{H} : A \times I \to X$ starting with f and ending with a function $\tilde{f}$ so that $\tilde{f}|_{A \cap B} = g|_{A \cap B}$. Now, since $\tilde{f}$ and g agree exactly on $A \cap B$, these functions glue to give a function $h : Y \to X$ so that $h|_A = \tilde{f}$ and $h|_B = g$. This is the required function. $\square$

What's interesting is that cohomology functors $\tilde{H}^n$ satisfy these same properties. I will demonstrate this to you, but first I need to state what exactly I mean by a functor that "satisfies these properties" and give such functors a name.

3. Definition of a Brown functor

Definition. A functor $T : \mathscr{C}^{op} \to \mathbf{Set}_*$ is called a **Brown functor** if

(1) (Wedge axiom) For any family of spaces $\{X_\alpha\}_{\alpha\in A}$ in $\mathscr{C}$, the natural map

$$T(\vee_{\alpha\in A} X_\alpha) \to \prod_{\alpha\in A} T(X_\alpha)$$

is an isomorphism.

Note: A corollary to the wedge axiom is that each Ti_β may be regarded as the projection map $\prod_{\alpha\in A} T(X_\alpha) \to T(X_\beta)$.

(2) (Mayer-Vietoris axiom)

If $X \to Y$ is an inclusion of spaces and $y \in T(Y)$, we will write $y|_X$ for the element $T(X \to Y)(y) \in T(X)$.

Note: In the case that $T = \mathrm{Hom}(-, Z)$, $T(X \to Y)(y)$ is literally the restriction of y to X, hence the notation.

If $(Y; A, B)$ is a CW-triad and $a \in T(A)$, $b \in T(B)$ are elements so that $a|_{A\cap B} = b|_{A\cap B}$, then there is an element $y \in T(Y)$ so that $y|_A = a$ and $y|_B = b$.

4. Cohomology is a Brown functor

Let $\tilde{H}^n$ be the degree n part of some reduced cohomology theory.

(1) (Wedge axiom) This is literally the additivity axiom for a cohomology theory:

$$\tilde{H}^n\left(\bigvee_{\alpha\in A} X_\alpha\right) \xrightarrow{\sim} \prod_{\alpha\in A} \tilde{H}^n(X_\alpha).$$

(2) (Mayer-Vietoris axiom) Given a CW-triad $(Y; A, B)$, consider the Mayer-Vietoris long exact sequence:

$$\cdots \to \tilde{H}^n(X) \to \tilde{H}^n(A) \oplus \tilde{H}^n(B) \to \tilde{H}^n(A\cap B) \to \cdots$$

(Diagram chase: the axiom follows directly from exactness in the middle here.)

This hints that cohomology functors and Hom functors have something to do with each other.