THE KÜNNETH THEOREM

Contents of these notes are based on

- *A Course in Homological Algebra*, P.J. Hilton, U. Stammbach

Preliminaries

The following results hold for modules over a PID, but the objects can be considered as abelian groups if desired. We begin by motivating a few definitions. Recall that for a module A, the functors $A \otimes -$ and $- \otimes A$ are always right exact, but not left exact (i.e. the functors don't preserve kernels). Consider the inclusion of $i : 2\mathbb{Z} \to \mathbb{Z}$ where both are considered modules over $\mathbb{Z}$. If we tensor with the $\mathbb{Z}$-module $\mathbb{Z}_2$, then the corresponding map $1 \otimes i : \mathbb{Z}_2 \otimes 2\mathbb{Z} \to \mathbb{Z}_2 \otimes \mathbb{Z}$ is now the zero map since the generator $1 \otimes 2 \in \mathbb{Z}_2 \otimes 2\mathbb{Z}$ can then factor as $1 \otimes 2 = 1 \otimes 2 \cdot 1 = 1 \cdot 2 \otimes 1 = 0 \in \mathbb{Z}_2 \otimes \mathbb{Z}$. In other words, the exact sequence

$$0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

is no longer exact when we tensor with $\mathbb{Z}_2$. This motivates the following definition.

<u>Definition:</u> A module P is called flat if the functor $P \otimes -$ is exact. That is, for any short exact sequence $0 \to A'' \to A \to A \to 0$, the sequence $0 \to P \otimes A'' \to P \otimes A \to P \otimes A \to 0$ is exact.

Over a PID it follows that a module is flat if and only if it is torsion free. Along these lines, we'll also need a "measure", so to speak, of the torsion in a module, or the "inexactness" of the functor $A \otimes -$. This comes from the Tor functor.

<u>Definition:</u> Given a module A and flat presentation $0 \to R \to P \xrightarrow{\varepsilon} A \to 0$ of A (i.e. short exact sequence with R and P flat), and any module B, we define $\mathrm{Tor}(A, B) := \mathrm{Ker}(R \otimes B \to P \otimes B)$, thus making the following sequence exact:

$$0 \to \mathrm{Tor}_\epsilon(A, B) \to R \otimes B \to P \otimes B \to A \otimes B \to 0$$

Analogously, let $0 \to S \to Q \xrightarrow{\eta} B \to 0$ be a flat presentation for B. We define $\overline{\mathrm{Tor}}_\eta(A, B) = \mathrm{Ker}(A \otimes S \to A \otimes Q)$, giving an exact sequence:

$$0 \to \overline{\mathrm{Tor}}_\eta(A, B) \to A \otimes S \to A \otimes Q \to A \otimes B \to 0$$

With a little effort, one can show that these two objects $\mathrm{Tor}_\varepsilon(A,B) \cong \overline{\mathrm{Tor}}_\eta(A,B)$ are naturally isomorphic and independent of the choice of flat presentation. So we may simply use the notation $\mathrm{Tor}(A,B)$ to mean the object with the above properties.

Lastly, we assume familiarity with chain complexes and recall that for two chain complexes $\mathbf{C}$ and $\mathbf{D}$ the tensor product is given by $(\mathbf{C}\otimes\mathbf{D})_n = \oplus_{p+q=n}\mathbf{C}_p \otimes \mathbf{D}_q$ with differentical $\partial^{\otimes} = \partial\otimes 1 + \eta\otimes\partial$ where $\eta(c) = (-1)^p$ for $c \in \mathbf{C}_p$. That is, for $c\otimes d \in \mathbf{C}_p \otimes \mathbf{D}_q$, $\partial^{\otimes}(c\otimes d) = \partial c \otimes d + (-1)^p c \otimes \partial d$.

The Künneth Formula

Theorem (Künneth): Let $\mathbf{C}$ and $\mathbf{D}$ be chain complexes over a PID, Λ, and suppose that at least one of $\mathbf{C}$ or $\mathbf{D}$ is flat (i.e. the modules in each degree are flat). Then for each n we obtain the following exact sequence:

$$0 \to \oplus_{p+q=n} H_p(\mathbf{C}) \otimes H_q(\mathbf{D}) \to H_n(\mathbf{C}\otimes\mathbf{D}) \to \oplus_{p+q=n-1}\mathrm{Tor}(H_p(\mathbf{C}), H_q(\mathbf{D})) \to 0$$

Proof. Without loss of generality suppose $\mathbf{C}$ is flat. We introduce the notation:

$$\begin{array}{ll} \mathbf{Z} = Z(\mathbf{C}) & \overline{\mathbf{Z}} = Z(\mathbf{D}) \\ \mathbf{B} = B(\mathbf{C}) & \overline{\mathbf{B}} = B(\mathbf{D}) \end{array}$$

We also introduce the notation $\mathbf{B}' = \mathbf{B}[-1]$, so that we may consider the differential $\partial : \mathbf{C} \to \mathbf{B}'$ as a chain map. We have an exact sequence:

$$0 \to \mathbf{Z} \xrightarrow{\iota} \mathbf{C} \xrightarrow{\partial} \mathbf{B}' \to 0.$$

Since $\mathbf{C}$ is flat and Λ is a PID, and $\mathbf{Z} \to \mathbf{C}$ is injective, it follows that $\mathbf{Z}$ is flat. Similarly we have $\mathbf{B}$ and $\mathbf{B}'$ are flat. Since $\mathbf{B}'$ is flat, we obtain the following short exact sequence of chains when tensoring with $\mathbf{D}$:

$$0 \to \mathbf{Z}\otimes\mathbf{D} \to \mathbf{C}\otimes\mathbf{D} \to \mathbf{B}'\otimes\mathbf{D} \to 0$$

which then gives us a long exact triangle in homology:

$$\begin{tikzcd} H(\mathbf{Z}\otimes\mathbf{D}) \arrow[rr, "\iota_*"] & & H(\mathbf{C}\otimes\mathbf{D}) \arrow[dl, "\partial_*"] \\ & H(\mathbf{B}'\otimes\mathbf{D}) \arrow[ul, dashed, "\omega"] & \end{tikzcd}$$

where the connection morphism ω is of degree -1 (denoted with a dashed arrow). We are going to adjust the diagram a bit by replacing $\mathbf{B}'$ with $\mathbf{B}$, making ∂_* a map of degree -1 and ω degree 0.

$$\begin{tikzcd}
H(\mathbf{Z}\otimes\mathbf{D}) \arrow[rr, "\iota_*"] & & H(\mathbf{C}\otimes\mathbf{D}) \arrow[dl, dashed, "\partial_*"] \\
& H(\mathbf{B}\otimes\mathbf{D}) \arrow[ul, "\omega"] &
\end{tikzcd}$$

Considering the chain complex $\mathbf{B}\otimes\mathbf{D}$, since the differential on $\mathbf{B}$ is trivial, we get that the differential $\partial^{\otimes} = \eta\otimes\partial$. So we can compute homology by considering $1\otimes\partial$.

$$\ldots \longrightarrow (\mathbf{B}\otimes\mathbf{D})_{n+1} \xrightarrow{(1\otimes\partial)_{n+1}} (\mathbf{B}\otimes\mathbf{D})_n \xrightarrow{(1\otimes\partial)_n} (\mathbf{B}\otimes\mathbf{D})_{n-1} \xrightarrow{(1\otimes\partial)_{n-1}} \ldots$$

Now since $\mathbf{B}$ is flat, it follows that $\ker(1\otimes\partial)_n = (\mathbf{B}\otimes\ker(\partial))_n = (\mathbf{B}\otimes\overline{\mathbf{Z}})_n$. We also have that $\operatorname{im}(1\otimes\partial)_{n+1} = \mathbf{B}\otimes\operatorname{im}(\partial)_{n+1} = (\mathbf{B}\otimes\overline{\mathbf{B}})_n$, so that $H_n(\mathbf{B}\otimes\mathbf{D}) \cong (\mathbf{B}\otimes H(\mathbf{D}))_n$. By the same arguments we find $H_n(\mathbf{Z}\otimes\mathbf{D}) \cong (\mathbf{Z}\otimes H(\mathbf{D}))_n$, so we may right the exact triangle:

$$\begin{tikzcd}
\mathbf{Z}\otimes H(\mathbf{D}) \arrow[rr, "\iota_*"] & & H(\mathbf{C}\otimes\mathbf{D}) \arrow[dl, dashed, "\partial_*"] \\
& \mathbf{B}\otimes H(\mathbf{D}) \arrow[ul, "\omega"] &
\end{tikzcd}$$

We wish to determine the behavior of the connection morphism ω, so by the standard construction we start with an element $\partial c\otimes[z] \in \mathbf{B}\otimes H(\mathbf{D})$, and then get $c\otimes z \in \mathbf{C}\otimes\mathbf{D}$ such that $(\partial\otimes 1)(c\otimes z) = \partial c\otimes z$. We then take $\partial^{\otimes}(c\otimes z) = \partial c\otimes z \in (\mathbf{C}\otimes\mathbf{D})$ and note that $(\partial\otimes 1)(\partial c\otimes z) = 0$, and by the exactness get the element $\partial c\otimes z \in \mathbf{Z}\otimes\mathbf{D}$. That is, ω is induced by the inclusion map $\mathbf{B}\to\mathbf{Z}$. Hence the $\operatorname{coker}(\omega) = H(\mathbf{C})\otimes H(\mathbf{D})$ and since $\iota\omega = 0$ we get a map $\Gamma : H(\mathbf{C})\otimes H(\mathbf{D}) \to H(\mathbf{C}\otimes\mathbf{D})$. Moreover, by the exactness of the triangle (i.e. $\operatorname{im}\omega = \ker\Gamma$) we get that Γ is injective.

$$\begin{tikzcd}
H(\mathbf{C})\otimes H(\mathbf{D}) \arrow[drr, "\Gamma"] & & \\
\mathbf{Z}\otimes H(\mathbf{D}) \arrow[u] \arrow[rr, "\iota_*"] & & H(\mathbf{C}\otimes\mathbf{D}) \arrow[dl, dashed, "\partial_*"] \\
& \mathbf{B}\otimes H(\mathbf{D}) \arrow[ul, "\omega"] &
\end{tikzcd}$$

Notice now that since $\mathbf{Z}$ and $\mathbf{B}$ are flat, we have a flat resolution of $H(\mathbf{C})$, so after tensoring with $H(\mathbf{D})$ we get that $\ker\omega = \operatorname{Tor}(H(\mathbf{C}), H(\mathbf{D}))$, together with a map from $H(\mathbf{C}\otimes\mathbf{D})$ since $\omega\partial_* = 0$.

$$\begin{tikzcd}
H(\mathbf{C}) \otimes H(\mathbf{D}) \arrow[rrd, "\Gamma"] & & \\
\mathbf{Z} \otimes H(\mathbf{D}) \arrow[u] \arrow[rr, "\iota_*"] & & H(\mathbf{C} \otimes \mathbf{D}) \arrow[ld, dashed, "\partial_*"] \arrow[d, dashed] \\
& \mathbf{B} \otimes H(\mathbf{D}) \arrow[lu, "\omega"] & \mathrm{Tor}(H(\mathbf{C}), H(\mathbf{D})) \arrow[l]
\end{tikzcd}$$

Lastly, by the exactness of the triangle, we get that the map $H(\mathbf{C} \otimes \mathbf{D}) \to \mathrm{Tor}(H(\mathbf{C}), H(\mathbf{D}))$ is surjective. Thus we obtain our exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H(\mathbf{C}) \otimes H(\mathbf{D}) \arrow[r, "\Gamma"] & H(\mathbf{C} \otimes \mathbf{D}) \arrow[r, dashed] & \mathrm{Tor}(H(\mathbf{C}), H(\mathbf{D})) \arrow[r] & 0
\end{tikzcd}$$

Explicitly:

$$0 \to \oplus_{p+q=n} H_p(\mathbf{C}) \otimes H_q(\mathbf{D}) \to H_n(\mathbf{C} \otimes \mathbf{D}) \to \oplus_{p+q=n-1} \mathrm{Tor}(H_p(\mathbf{C}), H_q(\mathbf{D})) \to 0.$$