This presentation follows the structure of the proof given in May's *A Concise Course in Algebraic Topology* and relies on the axioms for reduced homology.

Definition. From the dimension axiom for reduced homology, $\widetilde{H}_0(S^0) \cong \mathbb{Z}$. Fix i_0, a generator of $\widetilde{H}_0(S^0)$. The suspension axiom yields that $\widetilde{H}_{n+1}(S^{n+1}) \xrightarrow{\Sigma} \widetilde{H}_n(S^n) \cong \mathbb{Z}$ is an isomorphism for all $n \geq 0$. Let $i_{n+1} = \Sigma(i_n)$.
For a based topological space X, the *Hurewicz map*, $h : \pi_n(X) \to \widetilde{H}_n(X)$, is given by
$h([f]) = \widetilde{H}_n(f)(i_n)$.

Lemma. *The Hurewicz map is a natural group homomorphism.*

For $[f], [g] \in \pi_n(X)$, $[f] + [g] = [f + g]$ is given by the following composition of maps:

$$S^n \xrightarrow{\ p\ } S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\ \nabla\ } X.$$

where p is the pinch map and ∇ is the fold map.
Consider the composition

$$\widetilde{H}_*(S^n) \xrightarrow{\widetilde{H}_n(p)} \widetilde{H}_n(S^n \vee S^n) \xrightarrow{\widetilde{H}_n(f \vee g)} \widetilde{H}_n(X \vee X) \xrightarrow{\widetilde{H}_n(\nabla)} \widetilde{H}_n(X).$$

The image of i_n under this composite map is $\widetilde{H}_n(\nabla \circ f \vee g \circ p)(i_n) = \widetilde{H}_n(f + g)(i_n) = h([f + g])$.
From the additivity axioms, we have isomorphisms $\widetilde{H}_n(S^n) \oplus \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n \vee S^n)$ and
$\widetilde{H}_n(X) \oplus \widetilde{H}_n(X) \to \widetilde{H}_n(X \vee X)$.

$$\begin{tikzcd}
\widetilde{H}_*(S^n) \arrow[r, "\widetilde{H}_n(p)"] \arrow[dr, dashed] & \widetilde{H}_n(S^n \vee S^n) \arrow[r, "\widetilde{H}_n(f \vee g)"] & \widetilde{H}_n(X \vee X) \arrow[r, "\widetilde{H}_n(\nabla)"] & \widetilde{H}_n(X) \\
& \widetilde{H}_n(S^n) \oplus \widetilde{H}_n(S^n) \arrow[u, "\cong"] \arrow[r, dashed] & \widetilde{H}_n(X) \oplus \widetilde{H}_n(X) \arrow[u, "\cong"] \arrow[ur, dashed] &
\end{tikzcd}$$

Let the dashed arrows be the compositions that make the above diagram commute. The image of i_n along the dashed arrows is

$$i_n \mapsto (i_n, i_n) \mapsto (\widetilde{H}_n(f)(i_n), \widetilde{H}_n(g)(i_n)) \mapsto \widetilde{H}_n(f)(i_n) + \widetilde{H}_n(g)(i_n) = h([f]) + h([g])$$

Since the diagram commutes, h is a group homomorphism.
Let $f : X \to Y$ be a map of based spaces.

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\pi_n(f)"] \arrow[d, "h"'] & \pi_n(Y) \arrow[d, "h"] \\
\widetilde{H}_n(X) \arrow[r, "\widetilde{H}_n(f)"'] & \widetilde{H}_n(Y)
\end{tikzcd}$$

Note that $h \circ \pi_n(f)([g]) = h([f \circ g]) = \widetilde{H}_n(f \circ g)(i_n) = \widetilde{H}_n(f) \circ \widetilde{H}_n(g)(i_n) = \widetilde{H}_n(f) \circ h([g])$ for all $[g] \in \pi_n(X)$.
So the above diagram commutes for any f, i.e., h is natural.

Note. The suspension isomorphism Σ is also natural, so $\Sigma \circ h = h \circ \Sigma$.

Lemma. *For any CW-complex X, $\widetilde{H}_n(X) \cong \widetilde{H}_n(X^{n+1})$.*

Let $i \geq n + 1$. Since X^i is a subcomplex of X^{i+1}, there exists, by the exactness and suspension axioms for reduced homology, a long exact sequence:

$$\cdots \to \widetilde{H}_{n+1}(X^{i+1}/X^i) \to \widetilde{H}_n(X^i) \to \widetilde{H}_n(X^{i+1}) \to \widetilde{H}_n(X^{i+1}/X^i) \to \cdots$$

For any i, X^{i+1}/X^i is a wedge of $(i+1)$-spheres. So, $\widetilde{H}_n\left(X^{i+1}/X^i\right) \cong \bigoplus_{j\in I} \widetilde{H}_n(S^n_j)$ by additivity. By the suspension axiom, $\widetilde{H}_n(S^{i+1}) \cong \widetilde{H}_{n-1}(S^i) \cong \ldots \cong \widetilde{H}_{n-(i+1)}(S^0)$. Since $n-(i+1) \neq 0$, the dimension axiom yields that $\widetilde{H}_{n-(i+1)}(S^0) \cong 0$. So, $\widetilde{H}_n\left(X^{i+1}/X^i\right) \cong \bigoplus_{j\in I} 0$. An identical argument shows $\widetilde{H}_{n+1}\left(X^{i+1}/X^i\right) \cong 0$. So, the sequence below is exact:

$$0 \to \widetilde{H}_n\left(X^i\right) \to \widetilde{H}_n\left(X^{i+1}\right) \to 0$$

Thus, $\widetilde{H}_n\left(X^i\right) \cong \widetilde{H}_n\left(X^{i+1}\right)$ for all $i \geq n+1$. As a consequence, $\widetilde{H}_n\left(X^{n+1}\right) \cong \widetilde{H}_n\left(X^j\right)$ for all $j \geq n+1$.
So we have colim $\widetilde{H}_n(X^i) \cong \widetilde{H}_n(X^{n+1})$.

In *A Concise Course in Algebraic Topology*, May shows $H_n(X) = \text{colim } H_n(X_i)$ for any $X = X_0 \subseteq X_1 \subseteq \ldots$. The proof uses a construction, tel X_i, formed by attaching the mapping cyclinders for the inclusions $X_i \to X_{i+1}$. This construction is weakly equivalent to X, so $\widetilde{H}_n(\text{tel } X_i) \cong \widetilde{H}_n(X)$. Then the Mayer-Vietoris sequence for particular subspaces of tel X_i and an exact sequence for the colimit of abelian groups are used to show $\widetilde{H}_n(X) \cong \text{colim } \widetilde{H}_n(X_i)$. The proof depends on the additivity and weak equivalence axioms for homology of general topological spaces - for CW-complexes we only need additivity. See pages 114-116 in *Concise* for details.

Therefore, $\widetilde{H}_n(X) \cong \text{colim } \widetilde{H}_n(X^j) \cong \widetilde{H}_n(X^{n+1})$.

Lemma. *Let X be a wedge of n-spheres. Then $h : \pi_n(X) \to \widetilde{H}_n(X)$ is the abelianization homomorphism for $n = 1$ and an isomorphism for $n > 1$.*

If X is a single n-sphere, $\pi_n(X) \cong \mathbb{Z}\{[\text{id}]\}$ and $\widetilde{H}_n(X) \cong \mathbb{Z}\{i_n\}$ by the dimension and suspension axioms. Then $h([\text{id}]) = \widetilde{H}_n(\text{id})(i_n) = \text{id}(i_n) = i_n$, so $\mathbb{Z}\{[\text{id}]\} \xrightarrow{h} \mathbb{Z}\{i_n\}$ is an isomorphism. Note that since $\mathbb{Z}\{[\text{id}]\}$ is abelian, this also gives the conclusion for $n = 1$.
Now let $X = \bigvee_{j\in I} S^n_j$. By additivity, $\widetilde{H}_n(X) \cong \bigoplus_{j\in I} \widetilde{H}_n(S^n_j) \cong \bigoplus_{j\in I} \mathbb{Z}\{i_n\}$.
For $n > 1$, $\pi_n(X) = \mathbb{Z}\{I\} \cong \underset{j\in I}{\mathbb{Z}}\{[\text{id}]\}$. The map h is natural, $\pi_n(X)$ is generated by the inclusions $S^n_j \to X$, and the isomorphism $\bigoplus_{j\in I} \widetilde{H}_n(S^n_j) \to \widetilde{H}_n(X)$ is induced by the inclusions S^n_j, thus the following diagram commutes:

$$\begin{tikzcd}
\pi_n(S^n_j) \arrow[r] \arrow[d, "h"'] & \pi_n(X) \arrow[d, "h"] \\
\widetilde{H}_n(S^n_j) \arrow[r] & \widetilde{H}_n(X)
\end{tikzcd}$$

In particular, $h(\ldots, 0, [\text{id}], 0, \ldots) = (\ldots, h(0), h([\text{id}]), h(0), \ldots) = (\ldots, 0, i_n, 0, \ldots)$. Then h maps the k-th generator of of $\underset{j\in I}{\mathbb{Z}}\{[\text{id}]\}$ to the k-th generator of $\bigoplus_{j\in I} \mathbb{Z}\{i_n\}$ and is therefore an isomorphism.
In the case where $n = 1$, $\pi_n(X) \cong F^I$, the free group generated by the inclusions of the n-spheres into X. Then h maps the k-th generator of F^I to the k-th generator of $\bigoplus_{j\in I} \mathbb{Z}\{i_n\} = \left(F^I\right)^{ab}$ and is thus the abelianization homomorphism.

Theorem (Hurewicz). *Let X be any $(n-1)$-connected based space. Then the hurewicz homomorphism*

$$h : \pi_n(X) \to \widetilde{H}_n(X)$$

is the abelianization homomorphism for $n = 1$ and is an isomorphism for $n > 1$.

Proof. First, by the CW Approximation Theorem we may assume that X is a CW complex with a single 0-cell, based attaching maps, and no q-cells for $1 \leq q < n$. Further, the inclusion of the $(n+1)$ skeleton $X^{n+1} \hookrightarrow X$ induces

an isomorphism on the homotopy groups π_n. By the lemma above, we also have an isomorphism on the reduced homology groups $\widetilde{H}_n$. Therefore, it is no loss of generality to assume $X = X^{n+1}$.
We then have that X is the cofiber of a map $f : K \to L$ where $K = \bigvee_i S^n$ and $X^n = L = \bigvee_j S^n$:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d] & L \arrow[d] \\ \bigvee_i D^{n+1} \arrow[r] & X = C_f. \end{tikzcd}$$

Our goal is then to show that the top and bottom rows in the following commutative diagram are exact:

$$\begin{tikzcd} \pi_n(K) \arrow[r] \arrow[d, "h"] & \pi_n(L) \arrow[r] \arrow[d, "h"] & \pi_n(X) \arrow[r] \arrow[d, "h"] & \pi_{n-1}(K) = 0 \\ \widetilde{H}_n(K) \arrow[r] & \widetilde{H}_n(L) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & \widetilde{H}_{n-1}(K) = 0. \end{tikzcd}$$

We know the two vertical arrows on the left are isomorphisms for $n > 1$ and the abelianization homomorphism for $n = 1$ by the lemma, since K and L are wedges of n-spheres. So, showing exactness of the top and bottom rows will gives us that $h : \pi_n(X) \to \widetilde{H}_n(X)$ is an isomorphism (or abelianization homomorphism if $n = 1$).

First, we consider the sequence on the reduced homology groups. We have the cofiber sequence $K \xrightarrow{f} L \to X = C_f$, and would like to use the exactness axiom of reduced homology. Since K is not a subcomplex of L we cannot apply this directly. Instead we consider the mapping cylinder M_f,

$$\begin{tikzcd} & M_f \arrow[r] \arrow[d, "\simeq"', "r"] & M_f/K \arrow[d, "\cong"] \\ K \arrow[ur] \arrow[r, "f"] & L \arrow[r] & X = C_f \end{tikzcd}$$

Now K is a subcomplex of M_f so we have an exact sequence

$$\widetilde{H}_n(K) \to \widetilde{H}_n(M_f) \to \widetilde{H}_n(M_f/K) \to 0$$

But then since M_f/K is the cone C_f we have $M_f/K \cong C_f$ and since the retraction $r : M_f \to L$ is a homotopy equivalence we get the exact sequence we want:

$$\widetilde{H}_n(K) \to \widetilde{H}_n(L) \to \widetilde{X}_n \to 0.$$

Now for the sequence on homotopy groups, we need to consider two cases. First, assume $n > 1$. By a corollary of the homotopy excision theorem we know that for an n-equivalence $f : X \to Y$ where X is m-connected, the map $(M_f, X) \to (C_f, *)$ is an $(n + m + 1)$-equivalence. In our case we have that K is $(n - 1)$-connected, and the map $f : K \to L$ is an $(n - 1)$-equivalence, so $(M_f, K) \to (C_f, *) = (X, *)$ is a $(2n - 1)$-equivalence. Hence, we have an isomorphism

$$\pi_n(M_f, K) \to \pi_n(X, *)$$

for $n > 1$. Again considering the inclusion of K into the mapping cylinder and using the exact sequence on homotopy for the pair (M_f, K) we have

$$\begin{tikzcd}
\pi_n(K) \arrow[r] \arrow[d, equal] & \pi_n(M_f) \arrow[r] \arrow[d, "\simeq"] & \pi_n(M_f, K) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(K) = 0 \\
\pi_n(K) \arrow[r] & \pi_n(L) \arrow[r] & \pi_n(X) \arrow[r] & \pi_{n-1}(K) = 0.
\end{tikzcd}$$

Therefore, we have an exact sequence on the homotopy groups for $n > 1$. This gives us that the Hurewicz homomorphism is an isomorphism for $n > 1$.

Finally, for the case $n = 1$, we cannot use homotopy excision since that was only valid for $n > 1$. Let N denote the normal subgroup generated by $f(\pi_1(K))$ in $\pi_1(L)$.

In this case, from a corollary of the van Kampen theorem we have

$$\pi_1(X) = {}^{\pi_1(L)}/_{N}.$$

Therefore, we have an exact sequence

$$0 \longrightarrow N \longrightarrow \pi_1(L) \longrightarrow \pi_1(X) \longrightarrow 0.$$

Next, we claim that the abelianization N^{Ab} is isomorphic to the abelianization $f(\pi_1(K))^{\mathrm{Ab}}$. We can see this since if $x \in N^{\mathrm{Ab}}$ then there is some $y \in N$ such that $y \mapsto x$. But, $y \in N$ implies that $y = cf(k)c^{-1}$ for some $c \in \pi_1(L)$ and some $k \in \pi_1(K)$, so in N^{Ab} we have $x = f(k)$. This gives us a surjection $\pi_1(K) \xrightarrow{f^{\mathrm{Ab}}} N^{\mathrm{Ab}}$. Further, since abelianization is right exact we have

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{Ab}"] & \pi_1(L) \arrow[r] \arrow[d, "\mathrm{Ab}"] & \pi_1(X) \arrow[r] \arrow[d, "\mathrm{Ab}"] & 0 \\
& N^{\mathrm{Ab}} \arrow[r] & \pi_1(L)^{\mathrm{Ab}} \arrow[r] & \pi_1(X)^{\mathrm{Ab}} \arrow[r] & 0 \\
& \pi_1(K)^{\mathrm{Ab}} \arrow[r] \arrow[u, two heads] & \pi_1(L)^{\mathrm{Ab}} \arrow[r] \arrow[u, "\cong"] & \pi_1(X)^{\mathrm{Ab}} \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

Therefore, we get an exact sequence on the bottom row. Since the reduced homology groups $\widetilde{H}_1$ are abelian, the maps $\pi_1 \to \widetilde{H}_1$ factor through the abelianizations and we get a commutative diagram

$$\begin{tikzcd}
\pi_1(K) \arrow[r] \arrow[d, "\mathrm{Ab}"] \arrow[dd, bend right, dashed, "h"'] & \pi_1(L) \arrow[r] \arrow[d, "\mathrm{Ab}"] & \pi_1(X) \arrow[r] \arrow[d, "\mathrm{Ab}"] & 0 \\
\pi_1(K)^{\mathrm{Ab}} \arrow[r] \arrow[d, "\cong"] & \pi_1(L)^{\mathrm{Ab}} \arrow[r] \arrow[d, "\cong"] & \pi_1(X)^{\mathrm{Ab}} \arrow[r] \arrow[d, "\cong"] & 0 \\
\widetilde{H}_1(K) \arrow[r] & \widetilde{H}_1(L) \arrow[r] & \widetilde{H}_1(X) \arrow[r] & 0
\end{tikzcd}$$

We know that the two vertical compositions on the left are the abelianization homomorphisms from the lemma on wedges of n-spheres. Therefore, we have $h : \pi_1(X) \to \widetilde{H}_1(X)$ is the abelianization homomorphism, completing the proof. $\square$