OVERVIEW OF SPECTRA

Contents

1. Motivation

There are a lot of ways to motivate spectra. For example:

- Representing objects for generalized cohomology theories. Also give rise to generalized homology theories. We will do that next week.
- Exisive functors from $\mathcal{S}_*^{\text{fin}} \to \mathcal{S}_*$, where $\mathcal{S}_*$ is the infinity category of spaces. That perspective I'd like to get to, but it will be later in the semester and only if people are willing to work hard with me to do it.

For today, I'm going to take a different path, one which parallels last semester's topic of the derived category of an abelian category.

Last semester, we saw the following constructions. Let $\mathcal{A}$ be an abelian category, for example $\mathcal{A} = \text{R-Mod}$. We had the following picture.

$$
\begin{tikzcd}
 & & \mathcal{K}^-(\mathcal{P}) \arrow[r, "\cong"] \arrow[d] & \mathcal{D}^-(\mathcal{A}) \arrow[d] \\
\mathcal{A} \arrow[r] & Ch(\mathcal{A}) \arrow[r, "/\text{chain homot.}"'] \arrow[dr, "F", "\text{add. func}"'] & \mathcal{K}(\mathcal{A}) \arrow[r, "\text{invert q.isos.}"'] \arrow[d, "F"] & \mathcal{D}(\mathcal{A}) \arrow[dl, "\mathbf{R}F"', "\text{tot. derived func.}"] \\
 & & \text{Ab} &
\end{tikzcd}
$$

We saw that if $\mathcal{A}$ had enough projectives, then the subcategory $\mathcal{K}^-(\mathcal{P})$ of the bounded below chain complexes of projectives is equivalent to the derived category $\mathcal{D}^-(\mathcal{A})$. So in certain good situations, we don't have to invert quasi-isomorphisms (a potentially nasty process), to get the derived category we want to study.

Further, in many cases, these categories have a lot of structure. For example, $Ch(\mathcal{A})$ is often a closed symmetric moinoidal category (for example, if $\mathcal{A} =$ R-Mod), making it a great place to do mathematics.

In topology, things are a little backwards. Imagine that you knew what properties $\mathcal{D}(\mathcal{A})$ had, but didn't really know how to construct a good model for $Ch(\mathcal{A})$. You might get stuck trying to do algebra. For example, how do you put a module structure on cofibers? Let's make this problem precise. Look at what we do in an abelian context. Suppose that $\mathcal{A}$ is symmetric monoidal and that we have an algebra object in $\mathcal{A}$, say R, and let $x : R \to R$ be a self map. Then we can define R/x as the cokernel of x. We can produce a commutative diagram

$$\begin{tikzcd}
R \otimes R \arrow[r, "1\otimes x"] \arrow[d, "m"] & R \otimes R \arrow[r, "1\otimes p"] \arrow[d, "m"] & R \otimes R/x \arrow[r] \arrow[d, dotted, "!"] & 0 \\
R \arrow[r, "x"] & R \arrow[r] & R/x \arrow[r, "p"] & 0
\end{tikzcd}$$

and this gives R/x the structure of an R–module. Now, suppose that $\mathcal{D}(\mathcal{A})$ gets an induced symmetric monoidal structure and that we have an object in $R \in \mathcal{D}(\mathcal{A})$, . Our exact sequences are replaced by exact triangles, and R/x is the cofiber of x so that we have an exact triangle

$$R \xrightarrow{x} R \xrightarrow{p} R/x \to R[1].$$

Therefore, the best thing we can do is

$$\begin{tikzcd}
R \otimes R \arrow[r, "1\otimes x"] \arrow[d, "m"] & R \otimes R \arrow[r, "1\otimes p"] \arrow[d, "m"] & R \otimes R/x \arrow[r] \arrow[d, dotted] & R \otimes R[1] \arrow[d, "m[1]"] \\
R \arrow[r, "x"] & R \arrow[r] & R/x \arrow[r, "p"] & R[1]
\end{tikzcd}$$

but that arrow is not necessarily unique in $\mathcal{D}(\mathcal{A})$. So, it's not clear how to put a module structure on R/x.

For a long time, we've known what our derived category should looks like, but didn't have a good symmetric moinoidal category to represent it. This accounts for the strange history of the subject. So let's start by looking at this derived category, called the stable homotopy category.

2. Some recollections about $\mathbf{Top}_*$

Let $\mathbf{Top}_*$ be well-pointed pointed compactly generated weak Hausdorff topology spaces. Recall that a map $f : X \to Y$ is a weak homotopy equivalence if $\pi_0 f$ is a bijection and f it induces an isomorphism $\pi_*(X, x) \to \pi_*(Y, f(x))$ for all $x \in X$.

We have the following classical results:

Theorem 2.1 (Whitehead)**.** *A weak equivalence between* $\mathbf{CW}_*$*-complexes is a homotopy equivalence.*

Theorem 2.2 (CW-Approximation)**.** *There is a functor*

$$\Gamma : \mathbf{Top}_* \to \mathbf{CW}_*$$

and a natural transformation η *from* $\Gamma : \mathbf{Top}_* \to \mathbf{CW}_* \subseteq \mathbf{Top}_*$ *to* $\mathrm{id} : \mathbf{Top}_* \to \mathbf{Top}_*$*, and such that*

$$\Gamma X \xrightarrow{\eta_X} X$$

is a weak equivalence.

There are two "homotopy" categories that we can form from $\mathbf{Top}_*$. The classical one is $\mathbf{hTop}_*$, which is just spaces with homotopy classes of pointed maps. This would be the analogue of $\mathcal{K}(\mathcal{A})$. The second, we can form as follows.

Let $\mathbf{HoCW}_*$ be full subcategory of $\mathbf{hTop}_*$ whose objects are the CW-complexes.

Definition 2.3. Let $\mathbf{HoTop}_*$ be the category whose objects are topological spaces, but whose morphisms are

$$\mathrm{Hom}_{\mathbf{HoCW}_*}(\Gamma X, \Gamma Y) = [\Gamma X, \Gamma Y]_*.$$

Note that there is an equivalence of categories:

$$\mathbf{HoTop}_* \cong \mathbf{HoCW}_* .$$

By Whitehead's theorem, this is the same as "inverting" the weak homotopy equivalences. In other words, any functor

$$F : \mathbf{Top}_* \to \mathcal{C}$$

such that F takes weak homotopy equivalences to isomorphisms will factor through $\mathbf{HoTop}_*$.

3. Spanier Whitehead category

Now, note that $\mathbf{hTop}_*$ is not an algebraically flavored category. In general, $[X, Y]_*$ is just a pointed set. However, if $X = \Sigma X'$, then $[X, Y]_*$ is a group, and if $X = \Sigma X''$, then it's an abelian group.

To put ourselves in a more manageable setting, we might decide to turn $[X, Y]_*$ into abelian groups by lettting

$$\{X, Y\} = \mathrm{colim}_k [\Sigma^k X, \Sigma^k Y]_*$$

These are the stable homotopy classes of maps. If X and Y are finite CW complexes, then the Freudenthal Suspension theorem tells you that for $k \gg 0$,

$$[\Sigma^k X, \Sigma^k Y]_* \xrightarrow{\cong} [\Sigma^{k+1} X, \Sigma^{k+1} Y]_*$$

(in fact, all we need is $\dim(\Sigma^k X) \leq 2\text{dim-bottom-cell}(\Sigma^k Y) - 2$).

We can form a category whose objects are finite $\mathbf{CW}_*$-complexes $\mathbf{CW}_*^{\mathrm{fin}}$ and morphisms $\{X, Y\}$. Then

$$\Sigma : \mathbf{CW}_*^{\mathrm{fin}} \to \mathbf{CW}_*^{\mathrm{fin}}$$

is an isomorphism on hom-sets. However, it's not an equivalence of categories since it is not essentially surjective: not all spaces are the suspension of another space. However, we can remedy that by inverting Σ.

So we let $\mathcal{SW}$, the Spanier-Whitehead category, have objects (X, n) for X a finite pointed CW-complex and $n \in \mathbb{Z}$. Further,

$$\{X, Y\} = \mathrm{Hom}_{\mathcal{SW}}((X, m), (Y, n)) = \mathrm{colim}_k [\Sigma^{m+k} X, \Sigma^{n+k} Y]_*$$

Then, $\Sigma : \mathcal{SW} \to \mathcal{SW}$ is an automorphism. In fact, this category is

- Additive
- Triangulated
- Symmetric monoidal where

$$(X, m) \wedge (Y, n) = (X \wedge Y, n + m).$$

In fact, this is a model for the stable homotopy category of finite spectra, denote $\mathbf{HoSpectra}^{\mathrm{fin}}$. Further, there is a functor

$$\Sigma^\infty : \mathbf{HoCW}_*^{\mathrm{fin}} \to \mathbf{HoSpectra}^{\mathrm{fin}}$$

which sends X to $(X, 0)$. Now, we could forget the "finite CW-complex" part and try to build a larger Spanier-Whitehead category, but it won't have all coproducts and Σ^∞ won't be coproduct preserving. Indeed, consider

$$\left\{\left(\bigvee_{\alpha\in I} X_\alpha, 0\right), (Y,n)\right\} = \operatorname{colim}_k[\Sigma^k \bigvee_{\alpha\in I} X_\alpha, \Sigma^{k+n}Y]$$
$$= \operatorname{colim}_k \prod_{\alpha\in I}[\Sigma^k X_\alpha, \Sigma^{k+n}Y]$$

and we run into problems commuting limits and colimits.

What we would like is a diagram

$$\begin{tikzcd}
\mathbf{HoCW}_*^{\mathrm{fin}} \arrow[r] \arrow[d, "\Sigma^\infty"] & \mathbf{HoTop}_* \arrow[d, "\Sigma^\infty"] \\
\mathbf{HoSpectra}^{\mathrm{fin}} \arrow[r] & \mathbf{HoSpectra}
\end{tikzcd}$$

where Σ^∞ is colimit preserving functor.

4. Properties of the Stable Homotopy Category HoSpectra

I'm basing this section on C. Malkieviech, *The stable homotopy category* for this exposition.

There is a category, called the stable homotopy category, and denoted **HoSpectra**, with the following properties:

- **HoSpectra** is an additive category. That is, it has finite products and coproducts, is pointed (called the zero object $*$), the natural map

  $$X \vee Y \to X \times Y$$

  is an equivalence and

  $$[X, Y] := \mathbf{HoSpectra}(X, Y)$$

  are abelian groups.

  (Note, if X and Y are finite CW-complexes with bottom cell in high dimension so for example, if they are in the image of the suspension functor Σ^n for sufficiently large n, then $X \vee Y \to X \times Y$ is an isomorphism on homotopy groups in some range since they have the same k–skeleton up to a certain point.)
- There is a suspension functor $\Sigma : \mathbf{HoSpectra} \to \mathbf{HoSpectra}$ and a loop space $\Omega : \mathbf{HoSpectra} \to \mathbf{HoSpectra}$ such that Ω and Σ are inverse equivalences. In fact, with this functor, **HoSpectra** is triangulated.

- There is a functor $\Sigma^\infty : \mathbf{Top}_* \to \mathbf{HoSpectra}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{CW}_* \arrow[r] \arrow[d, "\Sigma"] & \mathbf{HoTop}_* \arrow[r, "\Sigma^\infty"] \arrow[d, "\Sigma"] & \mathbf{HoSpectra} \arrow[d, "\Sigma"] \\ \mathbf{CW}_* \arrow[r] & \mathbf{HoTop}_* \arrow[r, "\Sigma^\infty"] & \mathbf{HoSpectra} \end{tikzcd}$$

- Further, Σ^∞ as a right adjoint, $\Omega^\infty : \mathbf{HoSpectra} \to \mathbf{HoTop}_*$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r] \arrow[d, "\Omega"] & \mathbf{HoTop}_* \arrow[d, "\Omega"] & \mathbf{HoSpectra} \arrow[l, "\Omega^\infty"'] \arrow[d, "\Sigma"] \\ \mathbf{Top}_* \arrow[r] & \mathbf{HoTop}_* & \mathbf{HoSpectra} \arrow[l, "\Omega^\infty"] \end{tikzcd}$$

  Thus, we have

$$[\Sigma^\infty K, X] \cong \mathbf{HoTop}_*(K, \Omega^\infty X).$$

- (Cary had, if $A \to X$ has a retract, then $X = A \vee B$... but shouldn't this follow from triangulation axioms?)
- There is a special object in **HoSpectra**, the sphere spectrum

$$\mathbb{S} = \Sigma^\infty S^0.$$

  The homotopy groups of a spectrum are then defined as

$$\pi_n X = [\Sigma^n \mathbb{S}, X].$$

  Further, there is an isomorphism

$$\pi_n \Sigma^\infty K \cong \pi_n^s K = \operatorname{colim}_k \pi_{n+k}(\Sigma^k K).$$

- If a map $f : X \to Y$ in **HoSpectra** induces an isomorphisms on π_*, then it is an isomorphism in **HoSpectra**.
- **HoSpectra** is a closed symmetric monoidal category $(\mathbf{HoSpectra}, \wedge, \mathbb{S})$, where the internal hom is denoted by $F(X, Y)$. The "closed" means that there are isomorphisms

$$F(X \wedge Y, Z) \cong F(X, F(Y, Z)).$$

  Further, Σ^∞ is a moinoidal functor:

$$\Sigma^\infty(X \wedge Y) \cong \Sigma^\infty X \wedge \Sigma^\infty Y$$

in **HoSpectra**, and $\Sigma X = \Sigma^\infty S^1 \wedge X$ and $\Omega X = F(\Sigma^\infty S^1, X)$.

Definition 4.1. A ring spectrum is a monoid in **HoSpectra**.

Definition 4.2. Let A be an abelian group. An Eilenberg MacLane spectrum HR is a spectrum such that $\pi_n HR = R$ if $n = 0$ and 0 otherwise. If $A = R$ a ring, then HR is a ring spectrum.

Definition 4.3 (Pre-Definition)**.** Let **Spectra** be any category $\mathcal{C}$ with weak equivalences W such that the homotopy category $\mathcal{C}[W^{-1}]$ can be formed and is equivalent to **HoSpectra**.

Remark 4.4. Below, I'll give a first example, but note that later on, we will have closed symmetric monoidal models for **Spectra** in which we can do honest algebra. In there, for any ring spectrum E, we will be able to form the category of E–module spectra. In particular, if $E = HR$ (which will exist in our category), we will have the following result.

Theorem 4.5 (Shipley)**.** *Let R be a commutative ring. There is a triangulated equivalence between the categories* $\mathcal{D}$(R-Mod) *and the homotopy category of HR–modules.*

So, in particular, **Spectra** encompass classical homological algebra.

5. Topics

Here is a list of lectures I have in mind:

(1) Cohomology Theories and Brown Representability (Cherry and Sebastian)
(2) Prespectra and spectra
(3) The stable homotopy category as a triangulate symmetric monoidal category
(4) Interlude: Vector bundles, Thom Spaces, Thom Isomorphism theorem
(5) Thom Spectra, MO and MU, the Pontryagin-Thom theorem
(6) Quillen's theorem: MU and formal group laws
(7) K–theory, algebraic and topological (Markus)
(8) Interlude: Model Categories
(9) Symmetric, orthogonal and diagram Spectra
(10) $\mathbb{S}$–modules, E_∞ and A_∞–ring spectra (Agnes)
(11) The classical spectral sequences
(12) HR–module spectra (Shipley's theorem)
(13) Quotients and localization, and MU–module spectra (Morava K–theories)
(14) Interlude: Simplicial Sets
(15) ∞–categories
(16) Stable ∞–categories

(17) Spectrum objects and ∞–category of Spectra $\mathrm{Sp}(\mathcal{S}_*)$

(18) Symmetric monoidal structire on $\mathrm{Sp}(\mathcal{S}_*)$