HOMOLOGICAL ALGEBRA - LECTURE NOTES

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Abstract. These notes are based on the course Math 212, Homological Algebra given by professor Paul Balmer on Spring 2014. Most of these notes were live-TEXed in class. The footnotes are added by me.

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1. Abelian Categories

1.1. **Additive categories.**

Roughly, this means we can add morphisms $f+g$ and add objects $A \oplus B$.

Definition 1.1.1. An additive category is a category which satisfies the followings:

(1) there exists a zero object (final and initial)
(2) there exist a finite product & coproduct, and they are same ($A \coprod B \xrightarrow[\theta]{\sim} A \times B$)
(3) $\mathrm{Hom}(A,B)$ is an abelian group with induced operation, i.e., for $f,g : A \to B$

$$f+g \;:\; A \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} A \times A = A \coprod A \xrightarrow{\begin{pmatrix}f & 0\\0 & g\end{pmatrix}} B \times B = B \coprod B \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} B$$

Remark 1.1.2. The following maps are from universality.

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend left, "1"] & & \\
& A \coprod B \arrow[r, dashed, "\theta_A"] & A \\
B \arrow[ur] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A \\
A \coprod B \arrow[r, dashed, "\theta"] \arrow[urr, bend left, "\theta_A"] \arrow[drr, bend right, "\theta_B"'] & A \times B \arrow[ur] \arrow[dr] & \\
& & B
\end{tikzcd}$$

Definition 1.1.3. A category is preadditive if $\mathrm{Hom}(A,B)$ is abelian with bilinear composition, there is a zero object, and there is a biproduct $A \oplus B = A \times B = A \coprod B$ with four morphisms

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & & A \\
& A \oplus B \arrow[ur, "p_A"] \arrow[dr, "p_B"] & \\
B \arrow[ur, "i_B"] & & B
\end{tikzcd}$$

satisfying $p_A \circ i_A = id_A$, $i_A p_A + i_B p_B = id_{A \oplus B}$, etc.

Definition 1.1.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between (pre)additive categories. F is called additive if $F(f+g) = F(f) + F(g)$. This forces $F(A \oplus B) = F(A) \oplus F(B)$.[1]

Question 1.1.5. What are preadditive categories with only one object? [2]

Example 1.1.6. *R-Mod*, *R-Proj* (an R-module is projective if and only if it is a direct summand of a free module) and *R-Inj* are additive. Here *R-Proj* and *R-Inj* are full subcategories of projective, injective R-modules.

Example 1.1.7. If a category $\mathcal{A}$ is additive, then so is $\mathcal{A}^{op}$. We have $A^o \oplus B^o = (A \oplus B)^o$, $i_{A^o} = (p_A)^o$, etc.

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[1] By universality, we get the maps $F(A) \oplus F(B) \xrightarrow{\alpha} F(A \oplus B) \xrightarrow{\beta} F(A) \oplus F(B)$ and we can check that $\alpha = i_{F(A)} F(p_A) + i_{F(B)} F(p_B)$ and $\beta = F(i_A) p_{F(A)} + F(i_B) p_{F(B)}$. Thus we have $\alpha\beta = id_{F(A \oplus B)}$ and $\beta\alpha = id_{F(A) \oplus F(B)}$.

[2] We have only one datum - $\mathrm{Hom}(0,0)$ - which is an abelian group with bilinear compositions. Thus, each preadditive category corresponds to a ring, where the composition of morphisms corresponds to the multiplication of the ring.

Remark 1.1.8. Consider $f = (f_{ij})_{n \times m} : A_1 \oplus \cdots \oplus A_m \to B_1 \oplus \cdots \oplus B_n$ where $f_{ij} = p_{i,B} \circ f \circ i_{j,A}$. Composition of these maps corresponds to the matrix multiplication.

1.2. **Kernels and cokernels.**

Definition 1.2.1. Let $\mathcal{A}$ be an additive category and $f : A \to B$ be a morphism. We define the kernel of f by $(\ker f, i : \ker f \to A)$ if $fi = 0$ and it is a pullback (a limit), i.e., if $ft = 0$ below,

$$\begin{tikzcd}
T \arrow[drr, bend left, "0"] \arrow[dr, dashed, "\exists! \widetilde{t}"] \arrow[ddr, bend right, "t"'] & & \\
& \ker f \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

then there is a unique $\widetilde{t} : T \to \ker f$ satisfying $i\widetilde{t} = t$. Similarly, we can define the cokernel of f $(p : B \to \operatorname{coker} f)$ by a pushout:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "p"] \arrow[ddr, bend left, "s"] & \\
0 \arrow[r] \arrow[drr, bend right, "0"'] & \operatorname{coker} f \arrow[dr, dashed, "\exists! \widetilde{s}"] & \\
& & S
\end{tikzcd}$$

Definition 1.2.2. $f : A \to B$ is a monomorphism if $ft = ft'$ implies $t = t' : T \to A$, which is equivalent to say $\ker f = 0$. $f : A \to B$ is an epimorphism if $sf = s'f$ implies $s = s' : B \to S$, which is equivalent to say $\operatorname{coker} f = 0$.

Remark 1.2.3. If there is $\ker f$, then $i : \ker f \hookrightarrow A$ is a monomorphism.

Remark 1.2.4. The pullback of $\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$:

$$\begin{tikzcd}
T \arrow[drr, bend left] \arrow[dr, dashed] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & A \arrow[d] \\
& B \arrow[r] & C
\end{tikzcd}$$

exists if and only if $A \oplus B \xrightarrow{\begin{pmatrix} f & -g \end{pmatrix}} C$ has kernel $P \xrightarrow{\begin{pmatrix} f' \\ g' \end{pmatrix}} A \oplus B$.

Similarly, the pushout of $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & Q \arrow[dr, dashed] & \\
& & S
\end{tikzcd}$$

exists if and only if $A \to B \oplus C$ has cokernel.

Lemma 1.2.5. *Let* $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\curvearrowright"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$ *be a commutative diagram in an additive category. (We use notations* $\lrcorner$ *for pullback and* $\ulcorner$ *for pushout.)*

(1) *If this is cartesian (A is a pullback) and $\ker h$ exists, then $\ker g$ exists and $\ker g = \ker h$ in a compatible way with f, i.e., there is $i : \ker h \to A$, which is $\ker g$, and $fi = j : \ker h \to B$ is $\ker h$.*
(2) *dual statement holds for cocartesian (D is a pushout) case.*

Proof. Since A is a pullback, there is a unique $i : \ker h \to A$ induced by $hj = 0 = k \circ 0$

$$\begin{tikzcd} & \ker h \arrow[dl, dashed, "\exists! i"'] \arrow[d, "j"] \arrow[ddl, bend right=60, "0"'] \\ A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

We can check that $i : \ker h \to A$ is indeed the kernel of g. [3] ☐

1.3. **Abelian categories.**

Definition 1.3.1. An abelian category is an additive category in which every morphism has a kernel and a cokernel, and $\ker(\operatorname{coker}) = \operatorname{coker}(\ker)$:

$$\begin{tikzcd} & A \arrow[rrr, "f"] \arrow[dr, two heads] & & & B \arrow[dr, two heads, "p"] & \\ \ker f \arrow[ur, hook, "i"] & & \operatorname{coker} i \arrow[r, "\exists!", "\sim"'] \arrow[urr, dashed] & \ker p \arrow[ur, hook] & & \operatorname{coker} f \end{tikzcd}$$

The map is induced as follows. Since $fi = 0$, f factors through $A \twoheadrightarrow \operatorname{coker} i \to B$. Since the composition $pf : A \twoheadrightarrow \operatorname{coker} i \to B \twoheadrightarrow \operatorname{coker} f$ is zero and $A \twoheadrightarrow \operatorname{coker} i$ is an epimorphism, the composition $\operatorname{coker} i \to B \twoheadrightarrow \operatorname{coker} f$ is zero. Thus $\operatorname{coker} i \to B$ factors through $\operatorname{coker} i \to \ker p \hookrightarrow B$. We require that this induced map is an isomorphism.

Definition 1.3.2. Let $\mathcal{A}$ be an abelian category and $f : A \to B$ in $\mathcal{A}$. We define the image of f by

$$\operatorname{im} f = \operatorname{coker}(\ker f) = \ker(\operatorname{coker} f)$$

as seen in the factorization $\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & \operatorname{im} f \arrow[ur, hook] & \end{tikzcd}$.

Example 1.3.3. Consider $\mathbb{Z}$-*proj*, the full subcategory of finitely generated projective (=free) $\mathbb{Z}$-modules. $\mathbb{Z}$-*proj* is NOT an abelian category. Consider $f : \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ in $\mathbb{Z}$-*proj*. We have $\ker f = 0 = \operatorname{coker} f$, but the induced map $\mathbb{Z} = \operatorname{coker} i \xrightarrow{\times 2} \ker p = \mathbb{Z}$ is not an isomorphism.

[3]Given $t : T \to A$ such that $gt = 0$, there is a unique map $\tilde{t} : T \to \ker h$ such that $ft = j\tilde{t}$ by the definition of $\ker h$. We can check that $f(t - i\tilde{t}) = 0$ and $g(t - i\tilde{t}) = 0$, thus $t = i\tilde{t}$.

Question 1.3.4 (Final Problem #1). The category of Hausdorff topological abelian group (or $\mathbb{C}$-vector spaces) is not abelian even though all morphisms have kernels and cokernels. Cokernel is given by $\operatorname{coker}(f : V \to W) = W/\overline{\operatorname{im} f}$.

Remark 1.3.5. Kernels and cokernels are natural in the morphisms:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ A' \arrow[r, "g"] & B' \end{tikzcd} \Rightarrow \text{There exist } \widetilde{\alpha}, \widetilde{\beta} \text{ such that } \begin{tikzcd} \ker f \arrow[r] \arrow[d, "\widetilde{\alpha}"] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[d, "\widetilde{\beta}"] \\ \ker g \arrow[r] & A' \arrow[r] & B' \arrow[r] & \operatorname{coker} g \end{tikzcd}$$

commutes.

Proposition 1.3.6 (Epi-mono factorization)**.** *In an abelian category $\mathcal{A}$, all morphisms $f : A \to B$ factors uniquely and naturally as* $\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & \bullet \arrow[ur, hook] & \end{tikzcd}$.[4]

Proposition 1.3.7. *(1) A monomorphism is a kernel (of its cokernel).*
(2) An epimorphism is a cokernel (of its kernel).
(3) If a morphism is a monomorphism and an epimorphism, then it is an isomorphism.

Proof. If $f : A \to B$ is a monomorphism, then $\ker f = 0$, thus $A \xrightarrow{\sim} \operatorname{coker} f$. For (3), we have

$$\begin{tikzcd} A \arrow[rrr, "f"] \arrow[dr, "\sim"'] & & & B \\ & \operatorname{coker} i \arrow[r, "\sim"] & \ker p \arrow[ur, "\sim"'] & \end{tikzcd}$$

$\square$

Example 1.3.8. Let $\mathcal{C}$ be a small category (set of objects) and $\mathcal{A}$ be an abelian category. Define $\mathcal{A}^{\mathcal{C}} = \operatorname{Fun}(\mathcal{C}, \mathcal{A})$ be the category of functors and natural transformations. Then, $\mathcal{A}^{\mathcal{C}}$ is abelian with

$$\ker(\mathcal{F} : F \to G) : C \mapsto \ker(\mathcal{F}(C) : F(C) \to G(C)).$$

Example 1.3.9. Let $\mathcal{C}$ be an additive category and $\mathcal{A}$ be an abelian category. Then, $\operatorname{Add}(\mathcal{C}, \mathcal{A})$, the category of additive functors is abelian.

Example 1.3.10. Let X be a topological space and $\mathcal{A}$ be an abelian category. $PreSh_{\mathcal{A}}(X)$ with values in $\mathcal{A}$ is abelian with openwise kernel and cokernel. Indeed, $PreSh_{\mathcal{A}}(X) = \mathcal{A}^{Open(X)^{op}}$ where

$$\operatorname{Mor}_{Open(X)}(U, V) = \begin{cases} \varnothing & \text{if } U \not\subseteq V \\ U \hookrightarrow V & \text{if } U \subseteq V \end{cases}$$

Example 1.3.11. $Sh_{\mathcal{A}}(X)$ is abelian. Kernels are the ones in $PreSh_{\mathcal{A}}(X)$ and cokernels (or any colimits) are the sheafifications of the ones in $PreSh_{\mathcal{A}}(X)$.

Remark 1.3.12. A morphism $f : \mathcal{F} \to \mathcal{G}$ between sheaves is surjective if for all open $U \subseteq X$ and for all $b \in \mathcal{G}(U)$, there is a covering $U = \cup V_i$ and $a_i \in \mathcal{F}(V_i)$ such that $f(V_i)(a_i) = b|_{V_i}$ for all i. This is equivalent to say that $f_x : \mathcal{F}_x \twoheadrightarrow \mathcal{G}_x$ for all $x \in X$.

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[4] If we have another $\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & C \arrow[ur, hook] & \end{tikzcd}$, then we get the commutative diagram $\begin{tikzcd} \operatorname{coker} i \arrow[rr, "\sim"] \arrow[dr, dashed] & & \ker p \\ & C \arrow[ur, dashed] & \end{tikzcd}$.

1.4. **Exact sequences.**

Definition 1.4.1. The sequence $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0$ is exact at B if $\overline{f} : \operatorname{im} f \to \ker g$ is an isomorphism. [5]

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a short exact sequence if $f = \ker g$ and $g = \operatorname{coker} f$.

Exercise 1.4.2. $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $f = \ker g$. [6] Dually, $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if $g = \operatorname{coker} f$. Also, $0 \to \ker f \to A \xrightarrow{f} B \xrightarrow{g} C \to \operatorname{coker} f \to 0$ is exact.

Exercise 1.4.3. Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ and $gf = 0$. The followings are equivalent.

(1) The sequence is exact at B
(2) $\widetilde{f} : A \to \ker g$ is epic
(3) $\overline{g} : \operatorname{coker} f \to C$ is monic
(4) $0 \to \operatorname{im} f \to B \to \operatorname{im} g \to 0$ is a short exact sequence. [7]

Exercise 1.4.4. In $Sh_{\mathcal{A}}(X)$, the sequence $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ is exact if and only if $\mathcal{F}_x \xrightarrow{f_x} \mathcal{G}_x \xrightarrow{g_x} \mathcal{H}_x$ is exact for all $x \in X$.

Theorem 1.4.5 (Five Lemma)**.** *Suppose we have the following commutative diagram with exact rows:*

$$\begin{tikzcd}
A \arrow[r, "\alpha_1"] \arrow[d, "f"] & B \arrow[r, "\alpha_2"] \arrow[d, "g"] & C \arrow[r, "\alpha_3"] \arrow[d, "h"] & D \arrow[r, "\alpha_4"] \arrow[d, "i"] & E \arrow[d, "j"] \\
A' \arrow[r, "\beta_1"] & B' \arrow[r, "\beta_2"] & C' \arrow[r, "\beta_3"] & D' \arrow[r, "\beta_4"] & E'
\end{tikzcd}$$

If f, g, i, j are isomorphisms, then so is h.

Proof. Prove the special case first: if $A = A' = E = E' = 0$ and two of g, h, i are isomorphisms, then so is the other. Then derive the general case from

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im} \alpha_2 \arrow[r, hook] \arrow[d, "\wr"'] & C \arrow[r, two heads] \arrow[d] & \operatorname{im} \alpha_3 \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \operatorname{im} \beta_2 \arrow[r, hook] & C' \arrow[r, two heads] & \operatorname{im} \beta_3 \arrow[r] & 0
\end{tikzcd}$$

We get the two isomorphisms on the left and on the right by applying the special case successively on the left and on the right. $\square$

Remark 1.4.6. The above proof would be easier if we use element to chase around, i.e., when the abelian category admits a fully faithful functor $\mathcal{A} \to R\text{-}Mod$ such that a sequence in $\mathcal{A}$ is exact if and only if it is exact in $R\text{-}Mod$. This is true for a small (set of objects) abelian category by Freyd-Mitchell embedding theorem.

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[5]Consider $B \overset{p}{\twoheadrightarrow} \operatorname{coker} f$, $\ker g \overset{j}{\hookrightarrow} B$ and $\ker p \overset{u}{\hookrightarrow} B$. We also have an induced $\ker p \overset{\alpha}{\hookrightarrow} \ker g$. Then $(\operatorname{im} f \cong \ker g) \Leftrightarrow pj = 0 \Leftrightarrow (\operatorname{coker} f \cong \operatorname{im} g)$ by the following. If $\ker g \cong \operatorname{im} f = \ker p$, then clearly $pj = 0$. If $pj = 0$, then there exists $\ker g \xrightarrow{\beta} \ker p$ satisfying $u\beta = j$. By using $j\alpha = u$, we get $j\alpha\beta = u\beta = j$. Since j is monic, $\alpha\beta = 1$. Similarly we have $\beta\alpha = 1$, thus $\ker p \cong \ker g$.

[6]$\ker g = \operatorname{im} f = f$ since $0 \to A \xrightarrow{f} B$ is exact.

[7]For example, $0 \to \operatorname{im} f \to B \to \operatorname{im} g$ is exact if and only if $\operatorname{im} f = \ker(B \to \operatorname{im} g) = \ker g$.

Proposition 1.4.7. *Let $\mathcal{A}$ be an abelian category and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. The followings are equivalent:*

(1) f is a split monomorphism (i.e., there is $B \xrightarrow{r} A$ such that $rf = 1$.)
(2) g is a split epimorphism.
(3) The sequence is split exact (i.e., there are $B \xrightarrow{r} A, C \xrightarrow{s} B$ such that $rf = 1, gs = 1$ and $fr + sg = 1$.)
(4) There exists $h : B \to A \oplus C$ which makes the following commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "h", "\simeq"'] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "\binom{1}{0}"] & A \oplus C \arrow[r, "(0\ 1)"] & C \arrow[r] & 0 \end{tikzcd}$$

Proof. ((3) $\Rightarrow$ (1), (2)) and ((4) $\Rightarrow$ (3)) : Clear.
For (1) $\Rightarrow$ (4), use $h = \binom{r}{g}$ and use the five lemma. $\square$

Remark 1.4.8. In an abelian category, pushouts and pullbacks exist. For $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$, Consider $A \xrightarrow{\binom{-f}{g}} B \oplus C \xrightarrow{(h\ k)} D \to 0$. We can take $D = \operatorname{coker}(A \xrightarrow{\binom{-f}{g}} B \oplus C)$.

Definition 1.4.9. $\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$ is (co)cartesian if it is a pullback (pushout). It is bicartesian if both.

Proposition 1.4.10. *Consider the commutative diagram:* $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$. *The followings are equivalent:*

(1) It is bicartesian.
(2) $0 \to A \xrightarrow{\binom{-f}{g}} B \oplus C \xrightarrow{(h\ k)} D \to 0$ is exact.
(3) the induced maps $\widetilde{g} : \ker f \to \ker k$ and $\overline{h} : \operatorname{coker} f \to \operatorname{coker} k$ are isomorphisms.
(4) $\widetilde{f}$ and $\overline{k}$ are isomorphisms.

Proof. (1) $\Leftrightarrow$ (2) By the remark above.
(1) $\Rightarrow$ (3),(4) We've already seen that $\widetilde{f}$ is an isomorphism in the additive case.
(4) $\Rightarrow$ (1) By using $\mathcal{A}^{op}$, it is enough to show that it is cartesian. We need to show that for all $T \in \mathcal{A}$, there is a bijection

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{A}}(T, A) & \longleftrightarrow & \{(T \xrightarrow{s} B, T \xrightarrow{u} C) \mid hs = ku\} \\ t & \mapsto & (ft, gt) \end{array}$$

Suppose $ft = 0$ and $gt = 0$. Let $i : \ker g \hookrightarrow A$ and $j : \ker h \hookrightarrow B$. Then there exists $\widetilde{t} : T \to \ker g$ such that $t = i\widetilde{t}$. Since $j\widetilde{f}\widetilde{t} = fi\widetilde{t} = ft = 0$, we have $\widetilde{t} = 0$, i.e., $t = 0$.
On the other hand, consider $p : C \twoheadrightarrow \operatorname{coker} g$ and $q : D \twoheadrightarrow \operatorname{coker} h$. Since $\overline{k}$ is an isomorphism, we have $pu = \overline{k}^{-1}qku = \overline{k}^{-1}qhs = 0$. Take the epi-mono factorization of g, then u factors through

$\ker p = E$ via $\tilde{t} : T \to E$. Take a pullback P of x and $\tilde{t}$.

$$\begin{tikzcd}
 & {\color{red}P} \arrow[dl, two heads, red, "a"'] \arrow[ddr, red, "b"] & & \\
T \arrow[drrr, bend left, "s"] \arrow[ddr, dashed, "\tilde{t}"] \arrow[dddrr, bend right, "u"'] & & & \\
 & & A \arrow[r, "f"] \arrow[dl, two heads, "x"'] \arrow[dd, "g"'] & B \arrow[dd, "h"] \\
 & E \arrow[dr, hook, "y"] & & \\
 & & C \arrow[r, "k"] \arrow[d, two heads, "p"'] & D \arrow[d, two heads, "q"] \\
 & & \operatorname{coker} g \arrow[r, "\bar{k}", "\sim"'] & \operatorname{coker} h
\end{tikzcd}$$

We have $hfb = kgb = kyxb = ky\tilde{t}a = kua = hsa$, thus $h(fb - sa) = 0 = k(gb - ua)$. Now $0 \to P \to A \oplus T \to E \to 0$ is exact. [8] $\square$

Corollary 1.4.11. *Consider the commutative diagram:* $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "k"] & D \end{tikzcd}$.

(1) *Suppose this is cartesian. Then, f is monic if and only if k is monic. Suppose further that h is epic. Then, this is bicartesian and g is epic.*
(2) *Suppose this is cocartesian. Then, g is epic if and only if h is epic. Suppose further that f is monic. Then, this is bicartesian and k is monic.*

Proof. For (1), we have $\ker f \xrightarrow{\sim} \ker k$. If h is epic, then

$$0 \to A \to B \oplus C \to D \to 0$$

is exact. $\square$

Theorem 1.4.12 (Snake Lemma)**.** *Suppose we have the following commutative diagram with exact rows.*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

[8](??) need to fill in details!

Then, we have the long exact sequence given by the red line below:

$$
\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] & \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] \arrow[dddll, out=0, in=180, "\delta"] & & \\
& & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 & \\
& 0 \arrow[r] & A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] \arrow[d] & C' \arrow[d] & & \\
& & \operatorname{coker} a \arrow[r] & \operatorname{coker} b \arrow[r] & \operatorname{coker} c \arrow[r] & \operatorname{coker} g' \arrow[r] & 0
\end{tikzcd}
$$

This morphism δ is natural in the original data.

Question 1.4.13 (Final Problem #2). Prove Freyd-Mitchell embedding theorem or the snake lemma without using elements.

1.5. Functoriality in abelian categories.

Definition 1.5.1. An (additive) functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}, \mathcal{B}$ is exact if it preserves exact sequences.

Exercise 1.5.2. F preserves exact sequences
$\Leftrightarrow$ F preserves short exact sequences
$\Leftrightarrow$ F preserves kernels and cokernels.

Remark 1.5.3. If F preserves kernel (or cokernel), then it is automatically additive: $F(A \oplus B) = F(A) \oplus F(B)$.

Example 1.5.4. If $S \subseteq R$ is a multiplicative central subset ($S \subseteq Z(R), SS \subseteq S, 1 \in S$), then the functor $S^{-1}(-) : R\text{-}Mod \to (S^{-1}R)\text{-}Mod$ is exact.

Example 1.5.5. The sheafification functor $a : PreSh(X) \to Sh(X)$ is exact. However, the forgetful functor $u : Sh(X) \to PreSh(X)$ is not exact. Find an example! [9]

Definition 1.5.6. Let $\mathcal{B}$ be an abelian category. A subcategory $\mathcal{A} \subseteq \mathcal{B}$ is an abelian subcategory if $\mathcal{A}$ is abelian and $\mathcal{A} \hookrightarrow \mathcal{B}$ is exact. ($\Leftrightarrow$ sequences in $\mathcal{A}$ is exact if and only if they are exact in $\mathcal{B}$.)

Example 1.5.7. Let R be a ring and R-*mod* be the category of finitely generated R-modules. This is an abelian subcategory of R-*Mod* if and only if R is (left) noetherian. [10]

Exercise 1.5.8. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Then,

(1) F is faithful ($F(f) = 0 \Rightarrow f = 0$) if and only if F is conservative ($F(A) = 0 \Rightarrow A = 0$)

[9]Consider the sheaves $\mathcal{O}$ and $\mathcal{O}^\times$ on $\mathbb{C} \setminus \{0\}$ defined by the following : $\mathcal{O}(U)$ is the additive group of holomorphic functions on U and $\mathcal{O}^\times(U)$ is the multiplicative group of nonzero holomorphic functions on U. Consider the morphism $\exp : \mathcal{O} \to \mathcal{O}^\times$ which maps $f \in \mathcal{O}(U)$ to $e^{2\pi i f} \in \mathcal{O}^\times(U)$ for each $U \subseteq X$. Note that $\exp(\mathbb{C} \setminus \{0\})$ is not surjective because $z \in \mathcal{O}^\times(\mathbb{C} \setminus \{0\})$ is not in the image, but $\exp_x : \mathcal{O}_x \to \mathcal{O}_x^\times$ is surjective for each x, thus exp is surjective.
[10]A ring R is noetherian if and only if every submodule of finitely generated R-module is finitely generated.

(2) F is fully faithful, then F detects exactness. [11]

Definition 1.5.9. Let $\mathcal{A} \subseteq \mathcal{B}$ be a subcategory.
$\mathcal{A}$ is closed under subobjects if $B \hookrightarrow A \in \mathcal{A}$ in $\mathcal{B}$ implies $B \in \mathcal{A}$.
$\mathcal{A}$ is closed under quotients if $\mathcal{A} \ni A \twoheadrightarrow B$ in $\mathcal{B}$ implies $B \in \mathcal{A}$.
$\mathcal{A}$ is closed under extensions if $0 \to A \to B \to A' \to 0$ is a short exact sequence in $\mathcal{B}$ and $A, A' \in \mathcal{A}$, then $B \in \mathcal{A}$.
$\mathcal{A}$ is a Serre subcategory of $\mathcal{B}$ if it is a full, abelian subcategory closed under subobjects, quotients, and extensions.

Example 1.5.10. Let $S \subseteq R$ be a central multiplicative subset. Let S-*Tors* be the full subcategory of R-*Mod* such that $M \in S$-*Tors* if and only if $S^{-1}M = 0$. (S-*Tors* $= \ker(S^{-1}(-))$) This is a Serre subcategory of R-*Mod*.

Example 1.5.11. Let $F : \mathcal{B} \to \mathcal{C}$ be an exact functor between abelian categories. Then, $\ker(F)$ is a Serre subcategory of $\mathcal{B}$.

In fact, the converse also holds.

Theorem 1.5.12 (Gabriel, 1962)**.** *Let $\mathcal{A} \subseteq \mathcal{B}$ be a Serre subcategory of an abelian category $\mathcal{B}$. Then, there exists an exact functor $Q : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ to an abelian category $\mathcal{B}/\mathcal{A}$ which is initial (thus universal) among those $Q(\mathcal{A}) = 0$, i.e., for all exact functor $F : \mathcal{B} \to \mathcal{D}$ such that $F(\mathcal{A}) = 0$, there exists a unique functor $\overline{F} : \mathcal{B}/\mathcal{A} \to \mathcal{D}$ satisfying $\overline{F} \circ Q \simeq F$.*

Remark 1.5.13. $Q : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ is a (categorical) localization. Let

$$\mathcal{S} = \{f : B \to B' \text{ in } \mathcal{B} \mid \ker f, \operatorname{coker} f \in \mathcal{A}\}$$

Then for exact $F : \mathcal{B} \to \mathcal{D}$, we have $F(\mathcal{A}) = 0 \iff F(\mathcal{S}) \subseteq$ isomorphisms. Note that $(0 \to A)$ is in $\mathcal{S}$ for all $A \in \mathcal{A}$.

Remark 1.5.14. Strictly speaking, $\mathcal{B}/\mathcal{A}$ need to remain a category such that all $B \in \mathcal{B}$ have only sets of isomorphism classes of subobjects.

Proof of Theorem 1.5.12. (Gabriel construction of $\mathcal{B}/\mathcal{A}$)
Define $\operatorname{Obj}(\mathcal{B}/\mathcal{A}) = \operatorname{Obj}(\mathcal{B})$.
For $B, B' \in \mathcal{B}$, we define $\operatorname{Mor}_{\mathcal{B}/\mathcal{A}}(B, B')$ by the equivalence classes of

$$\begin{tikzcd} B \arrow[r, dashed] & B' \arrow[d, two heads, "\beta"] \\ X \arrow[u, hook, "\alpha"] \arrow[r, "f"] & Y \end{tikzcd}$$

[11] Firstly, we can show that a fully faithful exact functor F detects isomorphic objects. Suppose $F(A) \xrightarrow{\alpha} F(B)$ is an isomorphism in $\mathcal{B}$. Since F is full, there is $A \xrightarrow{f} B$ such that $F(f) = \alpha$. The sequence $0 \to \ker f \to A \xrightarrow{f} B$ is exact, thus so is $0 \to F(\ker f) \to F(A) \xrightarrow{\alpha = F(f)} F(B)$. Since α is monic, $F(\ker f) = 0$. Thus $\ker f = 0$ since F is conservative. Dually, we can show that f is epic, thus $A \cong B$ in $\mathcal{A}$. Now we only need to show that F detects kernels and cokernels. Suppose for given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$, we have the exact sequence $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ in $\mathcal{B}$. Let $\ker g \xrightarrow{j} B$. Since $F(f) = \ker F(g)$ and $F(g)F(j) = 0$, there's a morphism $F(\ker g) \to F(A)$. We can show that this is an inverse of the induced map $F(A \to \ker B)$, thus $f = \ker g$. Dually, we can do the same for cokernels.

such that $\operatorname{coker}\alpha, \ker\beta \in \mathcal{A}$. The equivalence relation is given by having common amplification:

$$\begin{tikzcd}
B \arrow[r, dashed] & B' \arrow[d, two heads, "\beta"] \\
X \arrow[u, hook, "\alpha"] \arrow[r] & Y \arrow[d, two heads, "\beta'"] \\
X' \arrow[u, hook, "\alpha'"] \arrow[r] & Y'
\end{tikzcd}$$

The composition is defined as follows:

$$\begin{tikzcd}
B \arrow[rrrr, dashed] & & & & B' \arrow[dl, two heads, "\beta"] \arrow[rrrr, dashed] & & & & B'' \\
& X \arrow[ul, hook, "\alpha"] \arrow[rr, "f"] & & Y & & X' \arrow[ul, hook, "\gamma"'] \arrow[ll, dashed, red, "(1)"'] \arrow[dl, two heads, red, "(2)"] \arrow[rr, "f'"] & & Y' \arrow[dl, two heads, blue, "\delta'"] \arrow[ur, two heads, "\delta"'] & \\
& & \color{blue}{X''} \arrow[ul, hook, blue, "\alpha'"] \arrow[rr, blue] & & \bullet \arrow[ul, hook, red, "(2)"] \arrow[rr, blue] & & \color{blue}{Y''} & &
\end{tikzcd}$$

where you

(1) compose $X' \to B' \to Y$
(2) get epi-mono factorization of (1)
(3) get a pullback X'' and a pushout Y''
(4) and compose $X'' \to \bullet \to Y''$.

Define $Q : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ by $\begin{tikzcd} B \arrow[r, dashed, "Q(f)"] \arrow[d, equal] & B' \arrow[d, equal] \\ B \arrow[r, "f"] & B' \end{tikzcd}$. Note that $Q(\overset{\alpha}{\hookrightarrow})$ is an isomorphism by $\begin{tikzcd} B' \arrow[r, equal] & B' \arrow[d, equal] \\ B \arrow[u, hook, "\alpha"] \arrow[r, hook, "\alpha"] & B' \end{tikzcd}$. $\square$

Remark 1.5.15. We say that an exact functor $F : \mathcal{B} \to \mathcal{C}$ is a quotient or a localization if you set $\mathcal{A} = \ker(F)$ or $\mathcal{S} = \{f \mid F(f) = 0\}$, then there is a unique map $\overline{F} : \mathcal{B}/\mathcal{A} \to \mathcal{C}$ such that $\overline{F} \circ Q \simeq F$ is an equivalence.

Example 1.5.16. Let R be a ring and $\mathcal{B} = R\text{-}Mod$. Let $S \subseteq R$ be a central multiplicative subset. Then $S^{-1} : R\text{-}Mod \to (S^{-1}R)\text{-}Mod$ is a quotient (i.e., localization) with respect to $t = \ker(S^{-1}(-)) = S\text{-}Tor$ the S-torsion R-modules. Indeed, we can identify $(S^{-1}R)\text{-}Mod$ with the full subcategory of $R\text{-}Mod$ on those $M \in R\text{-}Mod$ such that $s : M \to M, m \mapsto sm$ is an isomorphism for all $s \in S$. It is then easy to check the universal property for $S^{-1}(-)$.

Example 1.5.17. Let X be a space and consider $a : PreSh(X) \to Sh(X)$, the associative sheaf functor. This exact functor is a localization. Remember there is an adjunction: $\begin{tikzcd} PreSh(X) \arrow[d, shift right, "a"'] \\ Sh(X) \arrow[u, shift right, "u"'] \end{tikzcd}$

Remark 1.5.18. Recall that a pair of functors $\begin{tikzcd} \mathcal{C} \arrow[d, shift right, "F"'] \\ \mathcal{D} \arrow[u, shift right, "G"'] \end{tikzcd}$ are called adjoints if there exists a natural bijection

$$\alpha : \operatorname{Mor}_{\mathcal{D}}(F(x), y) \xrightarrow{\sim} \operatorname{Mor}_{\mathcal{C}}(x, G(y))$$

for $x \in \mathcal{C}, y \in \mathcal{D}$. It means to give "natural" transformations

$$\eta = \alpha(id_F) : Id_{\mathcal{C}} \to GF, \qquad \epsilon = \alpha^{-1}(id_G) : FG \to Id_{\mathcal{D}}$$

(η : unit, ϵ : counit) [12] such that $\begin{tikzcd} F \arrow[r, "F(\eta)"] \arrow[rr, bend right, "id"'] & FGF \arrow[r, "\epsilon_F"] & F \end{tikzcd}$ and $\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[rr, bend right, "id"'] & GFG \arrow[r, "G(\epsilon)"] & G \end{tikzcd}$. [13]

Conversely, to recover α,

$$\begin{array}{ccccc} \mathrm{Mor}_{\mathcal{D}}(F(x), y) & \stackrel{G}{\to} & \mathrm{Mor}_{\mathcal{C}}(GF(x), G(y)) & \xrightarrow{-\circ\eta} & \mathrm{Mor}_{\mathcal{C}}(x, G(y)) \\ (f : F(x) \to y) & \mapsto & G(f) & \mapsto & G(f) \circ \eta =: \alpha(f) \end{array}$$

Similarly, $\alpha^{-1}(g) := \epsilon \circ F(g)$ [14].

Remark 1.5.19. In particular, if $\mathcal{C}, \mathcal{D}$ are (pre)additive, and F, G are additive, then α is automatically an isomorphism of abelian groups (i.e., $\mathbb{Z}$-linear).

Proposition 1.5.20. *Let* $\begin{tikzcd} \mathcal{B} \arrow[d, shift right, "Q"'] \\ \mathcal{C} \arrow[u, shift right, "R"'] \end{tikzcd}$ *be an adjunction of additive functors between abelian categories. Suppose Q is exact and R is fully faithful. Then, Q is a Gabriel quotient (i.e., localization).*

Proof. For $c, c' \in \mathcal{C}$, one checks that the composite isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(c, c') \xrightarrow[\text{fully faithful}]{R} \mathrm{Hom}_{\mathcal{B}}(R(c), R(c')) \xrightarrow[adj]{\sim} \mathrm{Hom}_{\mathcal{C}}(QR(c), c')$$

is given by precomposition with $\epsilon_c : QR(c) \to c$. Hence by Yoneda, ϵ_c is an isomorphism for all $c \in \mathcal{C}$ [15].
By the unit-counit relation ($\epsilon_Q \circ Q(\eta) = id$), it follows that $Q(\eta_b)$ is an isomorphism for all $b \in \mathcal{B}$. In other words, since Q is exact, $\eta_b : b \to RQ(b)$ has kernel and cokernel in $\mathcal{A} := \ker(Q)$. Let us prove the universal property: $\begin{tikzcd} \mathcal{B} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{D} \\ \mathcal{C} \arrow[ur, dashed, "\exists \overline{F}"'] \end{tikzcd}$. Let $F : \mathcal{B} \to \mathcal{D}$ be exact such that $F(\mathcal{A}) = 0$.

Then, $F(\eta_b)$ is an isomorphism for all $b \in \mathcal{B}$. Thus we have $F(\eta) : F(id_{\mathcal{B}}) \xrightarrow{\cong} FRQ$. Let $\overline{F} : \mathcal{C} \to \mathcal{D}$ to be $\overline{F} = FR$, then we have $\overline{F}Q \simeq F$. Uniqueness is clear from $\epsilon : QR \cong id_{\mathcal{C}}$ ($\overline{F} \circ Q \cong \widetilde{F} \circ Q \stackrel{-\circ R}{\Longrightarrow} \overline{F} \cong \widetilde{F}$). $\square$

[12] naturality of η and ϵ is from that of α.

[13] For example, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FGF(x), F(x)) & \xrightarrow[\sim]{\alpha} & \mathrm{Hom}_{\mathcal{C}}(GF(x), GF(x)) \\ {\scriptstyle -\circ F(\eta_x)}\downarrow & & \downarrow{\scriptstyle -\circ\eta_x} \\ \mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) & \xrightarrow[\sim]{\alpha} & \mathrm{Hom}_{\mathcal{C}}(x, GF(x)) \end{array}$$

From this, we get $(\epsilon_F \circ F(\eta))(x) = \alpha^{-1}(id_{GF(x)}) \circ F(\eta_x) = id_{F(x)}$.

[14] For $F(x) \xrightarrow{f} y$, we have $\alpha^{-1}\alpha(f) = \epsilon_y \circ FG(f) \circ F(\eta_x) = f \circ \epsilon_{F(x)} \circ F(\eta_x) = f \circ (\epsilon_F \circ F(\eta))(x) = f$ by naturality of ϵ.

[15] By Yoneda's lemma, we have $\mathrm{Hom}_{COF}(\mathrm{Hom}_{\mathcal{C}}(c, -), \mathrm{Hom}_{\mathcal{C}}(QR(c), -)) \simeq \mathrm{Hom}_{\mathcal{C}}(QR(c), c) \ni \epsilon_c$.

Example 1.5.21. Let $U \subseteq X$ open, and let $j : U \hookrightarrow X$ the inclusion. Consider $\begin{array}{c} Sh(X) \\ {\scriptstyle j^*=\mathrm{res}_U}\big\downarrow \\ Sh(U) \end{array}$. res_U is exact. It has a right adjoint $j_* : Sh(U) \to Sh(X)$ defined by $j_*\mathcal{G}(V) = \mathcal{G}(U \cap V)$. (No sheafification needed) Note that $j^*j_* \xrightarrow{\sim} id$. Hence j_* is faithful. It is also fully faithful. Hence res_U is a localization.

Exercise 1.5.22. Write the adjunction in detail! [16]

1.6. **Left and right exact functors.**

Remark 1.6.1. Many functors between abelian categories are only partially exact.

(1) $M \otimes_R - : R\text{-}Mod \to Ab$ does not preserve monomorphisms, unless M is flat.
(2) $\mathrm{Hom}_R(M, -) : R\text{-}Mod \to Ab$ does not preserve epimorphisms, unless M is projective.
(3) $\mathrm{Hom}_R(-, M) : (R\text{-}Mod)^{op} \to Ab$ does not send all monomorphisms to epimorphisms, unless M is injective.
(4) $\Gamma(X, -) : Sh(X) \xrightarrow[\mathcal{F} \mapsto \mathcal{F}(X)]{} Ab$ does not preserve epimorphisms.

Exercise 1.6.2 (Tor-teaser)**.** Prove that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, then $0 \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$ is exact if M_3 is flat. [17]

Definition 1.6.3. A (additive) functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is left exact if it preserves kernels ($F(\ker f) \cong \ker F(f)$). It is right exact if it preserves cokernels.
A contravariant functor $F : \mathcal{A} \to \mathcal{B}$ is said to have those properties when considered as (covariant) $\mathcal{A}^{op} \to \mathcal{B}$.

Example 1.6.4. $F : \mathcal{A}^{op} \to \mathcal{B}$ is left exact if $A_1 \to A_2 \to A_3 \to 0$ exact $\Rightarrow$ $0 \to F(A_3) \to F(A_2) \to F(A_1)$ exact.

Proposition 1.6.5. *(1) $F : \mathcal{A} \to \mathcal{B}$ is left exact if and only if for every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$, the sequence $0 \to F(A_1) \to F(A_2) \to F(A_3)$ is exact.*
(2) $F : \mathcal{A} \to \mathcal{B}$ is right exact if and only if for every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$, the sequence $F(A_1) \to F(A_2) \to F(A_3) \to 0$ is exact.
(3) $F : \mathcal{A}^{op} \to \mathcal{B}$ is left exact if for every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$, the sequence $0 \to F(A_3) \to F(A_2) \to F(A_1)$ is exact.
(4) $F : \mathcal{A}^{op} \to \mathcal{B}$ is right exact if for every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$, the sequence $F(A_3) \to F(A_2) \to F(A_1) \to 0$ is exact.

Proof. Exercise! □

Remark 1.6.6. The goal of so-called "Derived Functors" is to provide a measure of failure of exactness.

[16] We define maps $\mathrm{Hom}_{Sh(U)}(j^*\mathcal{F}, \mathcal{G}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{Sh(X)}(\mathcal{F}, j_*\mathcal{G})$ by $\alpha(\phi)(V) = \phi(V \cap U) \circ \mathrm{res}_{V, V\cap U}$ for $V \subseteq X$ open, and $\beta(\psi)(W) = \psi(W)$ for $W \subseteq U$ open. We can easily see that α and β are inverses.
[17] An R-module is flat if and only if it is a direct limit of finitely generated free modules. See also 2.4.10.

Proposition 1.6.7. *Let* $\begin{tikzcd} \mathcal{C} \arrow[d, shift right, "F"'] \\ \mathcal{D} \arrow[u, shift right, "G"'] \end{tikzcd}$ *be an adjunction of functors between abelian categories. Then the left adjoint F is right exact, and the right adjoint G is left exact (and they are additive).*

Proof. In any adjunction of categories, the left adjoint preserves those colimit which exist in $\mathcal{C}$, and the right adjoint preserves those limits which exist in $\mathcal{D}$. Indeed,

$$\begin{aligned} \mathrm{Mor}_{\mathcal{D}}(F(\varinjlim x_i), y) &\cong \mathrm{Mor}_{\mathcal{C}}(\varinjlim x_i, G(y)) \cong \varprojlim \mathrm{Mor}_{\mathcal{C}}(x_i, G(y)) \\ &\cong \varprojlim \mathrm{Mor}_{\mathcal{D}}(F(x_i), y) \cong \mathrm{Mor}_{\mathcal{D}}(\varinjlim F(x_i), y) \end{aligned}$$

Hence F preserves coproduct (hence $\oplus$, hence F is additive), 0 (as an empty colimit), and pushouts, e.g., that of $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d] & \cdot \\ 0 & \end{tikzcd}$, i.e., cokernels. So F is right exact. ∎

Example 1.6.8. (1) $\begin{tikzcd} PreSh(X) \arrow[d, shift right, "a"'] \\ Sh(X) \arrow[u, shift right, "u"'] \end{tikzcd}$: a is left exact [18], hence a is exact.

(2) $\begin{tikzcd} Sh(X) \arrow[d, shift right, "j^*"'] \\ Sh(U) \arrow[u, shift right, "j_*"'] \end{tikzcd}$ where $U \overset{j}{\hookrightarrow} X$, open : j^* is left exact, hence j^* is exact. [19]

(3) $\begin{tikzcd} R\text{-}Mod \arrow[d, shift right, "M\otimes_R -"'] \\ S\text{-}Mod \arrow[u, shift right, "\mathrm{Hom}_S(M,-)"'] \end{tikzcd}$ for ${}_SM_R$: $M \otimes_R -$ is right exact, $\mathrm{Hom}_S(M, -)$ is left exact. [20]

(4) (??) $\begin{tikzcd} Mod\text{-}R \arrow[d, shift right, "\mathrm{Hom}_R(-,N)"'] \\ S\text{-}Mod \arrow[u, shift right, "\mathrm{Hom}_S(-,N)"'] \end{tikzcd}$ for ${}_SN_R$: (check this!) $\mathrm{Hom}_R(-, N)$ is right exact, BUT as a functor $Mod\text{-}R \to (S\text{-}Mod)^{op}$ it is left exact $(Mod\text{-}R)^{op} \to S\text{-}Mod$. [21]

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[18] a preserves kernel because a presheaf kernel is a sheaf.

[19] In general, if we have a morphism $f : X \to Y$, we have the adjunction $\begin{tikzcd} Sh(Y) \arrow[d, shift right, "f^{-1}"'] \\ Sh(X) \arrow[u, shift right, "f_*"'] \end{tikzcd}$ where f^{-1} is the sheafification of the presheaf $f^{-1}\mathcal{G}(U) = \varinjlim_{f(U)\subseteq V} \mathcal{G}(V)$.

[20] This is from $\mathrm{Hom}_S({}_SM_R \otimes_R {}_RN, {}_SN') \cong \mathrm{Hom}_R({}_RN, \mathrm{Hom}_S({}_SM_R, {}_SN'))$. Note that we have ${}_SA_R \otimes_R {}_RB \in S\text{-}Mod$, $C_R \otimes_R {}_RD_S \in Mod\text{-}S$, $\mathrm{Hom}_R({}_RC_S, {}_RD) \in S\text{-}Mod$, $\mathrm{Hom}_R({}_RA, {}_RB_S) \in Mod\text{-}S$.

[21] More examples are : $\begin{tikzcd} Sets \arrow[d, shift right, "\mathrm{free}"'] \\ Gp \arrow[u, shift right, "\mathrm{forget}"'] \end{tikzcd}$, $\begin{tikzcd} Gp \arrow[d, shift right, "\Delta(\mathrm{diagonal})"'] \\ Gp^2 \arrow[u, shift right, "\prod"'] \end{tikzcd}$, $\begin{tikzcd} Ab^2 \arrow[d, shift right, "\oplus"'] \\ Ab \arrow[u, shift right, "\Delta"'] \end{tikzcd}$, $\begin{tikzcd} Gp \arrow[d, shift right, "-/[-,-]"'] \\ Ab \arrow[u, shift right] \end{tikzcd}$ and $\begin{tikzcd} G\text{-}Mod \arrow[d, shift right, "\mathrm{forget}"'] \\ H\text{-}Mod \arrow[u, shift right, "\mathrm{Ind}_H^G(-)"'] \end{tikzcd}$ for groups $H \leq G$.

1.7. **Injectives and projectives.**

Let $\mathcal{A}$ be an abelian category throughout this section.

Definition 1.7.1. An object I in $\mathcal{A}$ is injective if $\mathrm{Hom}(-,I):\mathcal{A}^{op}\to Ab$ is exact. An object P in $\mathcal{A}$ is projective if $\mathrm{Hom}(P,-):\mathcal{A}\to Ab$ is exact. Since both functors are always left exact, we have the "usual" definition:
I injective
$\Leftrightarrow$ for all $M \overset{\alpha}{\hookrightarrow} N$ and for all $f: M\to I$, there is $\widetilde{f}: N\to I$ such that $\widetilde{f}\circ\alpha = f$

$\Leftrightarrow$ $\begin{tikzcd} M \arrow[r, "\forall"] \arrow[d, hook, "\forall"'] & I \\ N \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$: I has the "right lifting property" with respect to monomorphisms.

Dually,
P projective
$\Leftrightarrow$ for all $M \overset{\beta}{\twoheadrightarrow} N$ and for all $g: P\to N$, there is $\widetilde{g}: P\to M$ such that $\beta\circ\widetilde{g} = g$

$\Leftrightarrow$ $\begin{tikzcd} & M \arrow[d, two heads, "\forall"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"] & N \end{tikzcd}$: P has the "left lifting property" with respect to epimorphisms.

Example 1.7.2. In R-Mod, an object is projective if and only if it is a direct summand of a free module. Indeed,

(1) free modules $F = R^{(B)}$ for a set B $\left(f = \sum_{b\in B} f_b \mathbf{e}_b \in F\right)$ are projective:

$$\mathrm{Hom}_{R\text{-}Mod}(R^{(B)}, M) = \mathrm{Mor}_{Sets}(B, M), \qquad \begin{tikzcd} Sets \arrow[d, shift right, "F=R^{(-)}"'] \\ R\text{-}Mod \arrow[u, shift right, "U"'] \end{tikzcd}$$

(2) every R-module M is a quotient of a free module :

$$F(U(M)) = R^{(M)} \xrightarrow[\mathbf{e}_m\mapsto m]{} M$$

(3) the following useful general fact.

Proposition 1.7.3. *(1) If $F \overset{\beta}{\twoheadrightarrow} P$ is an epimorphism and P is projective, then β is a split epimorphism.*

(2) If $I \overset{\alpha}{\hookrightarrow} N$ is a monomorphism and I is injective, then α is a split monomorphism.

(3) If $M_1 \overset{\alpha}{\hookrightarrow} M_2 \overset{\beta}{\twoheadrightarrow} M_3$ is a short exact sequence and M_1 is injective or M_3 is projective, then the sequence is split exact. (hence the image of the sequence remains exact under any additive functor.)

Proof. (1) Look at the following: $\begin{tikzcd} & F \arrow[d, two heads] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, equal] & P \end{tikzcd}$

(2) Do the case(1)'s op.

(3) (1)+(2). $\square$

Proposition 1.7.4. *A (left) R-module I is injective if and only if it has the right lifting (i.e., the extension) property with respect to the monomorphisms of the form $J\hookrightarrow R$ for J (left) ideal in R.*

Proof. This is necessary.

$$\begin{tikzcd} J \arrow[r, "\forall"] \arrow[d, hook] & I \\ R \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Suppose I has this property. Let $M \hookrightarrow N$ be an arbitrary monomorphism of R-modules and $f : M \to I$ a homomorphism.
By Zorn's lemma, there exists $M \subseteq M' \subseteq N$ and $f' : M' \to I$ such that $f'|_M = f$ and which is maximal among extensions (obvious sense). We have to show that $M' = N$. So we're back to initial question but we can assume that M is maximal.
Suppose $M \neq N$, and let $m \in N \setminus M$. It suffices to show that

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook] & I \\ M + Rm \arrow[ur, dashed] & \end{tikzcd}$$

there is an extension of f to $M + Rm$ to get a contradiction. Let $J = Ann_R(m)$ and consider the following:

$$\begin{tikzcd} Rm \cap M \arrow[r, hook] \arrow[d, hook] & M \arrow[r, "f"] \arrow[d] & I \\ R/J \cong Rm \arrow[r] & M + Rm \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

Since this is a pushout, the existence of $\widetilde{f}$ follows if I has the extension property with respect to $Rm \cap M \hookrightarrow Rm$. Note that for some ideal $J \subseteq J' \subseteq R$, we have $J'/J \cong Rm \cap M$, thus

$$\begin{tikzcd} J' \arrow[r, two heads] \arrow[d, hook] & J'/J \arrow[r] \arrow[d, hook] & I \\ R \arrow[r, two heads] & R/J \arrow[ur, dashed] & \end{tikzcd}$$

Note that this is a pushout again. So the extension property boils again to the extension property with respect to $J' \hookrightarrow R$. $\square$

Corollary 1.7.5. *An abelian group I is injective (in $\mathcal{A} = \mathbb{Z}$-Mod) if and only if it is divisible, i.e., for all $x \in I$ and all $n \neq 0$, there exists $y \in I$ such that $ny = x$.*

Proof. Do the extension property with respect to $n\mathbb{Z} \hookrightarrow \mathbb{Z}$. $\square$

Definition 1.7.6. $\mathcal{A}$ has enough projectives if for every object $A \in \mathcal{A}$, there exists projective P and an epimorphism $P \twoheadrightarrow A$.
$\mathcal{A}$ has enough injectives if for every $A \in \mathcal{A}$, there exists injective I and a monomorphism $A \hookrightarrow I$.

Exercise 1.7.7. An arbitrary product of injectives is injective, and an arbitrary coproduct of projectives is projective. [22]

Proposition 1.7.8. *Let M be an abelian group. Then, $M \xrightarrow[m \mapsto (f(m))_f]{} \prod_{f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$ is a monomorphism into an injective. Hence, $\mathbb{Z}$-Mod $= Ab$ has enough injectives.*

[22]A product of exact functors is exact.

Proof. Since $\mathbb{Q}/\mathbb{Z}$ is divisible, thus $\prod \mathbb{Q}/\mathbb{Z}$ is injective. Now it is enough to show that α is a monomorphism. We can show that for all $0 \neq m \in M$, there exists $f : M \to \mathbb{Q}/\mathbb{Z}$ such that $f(m) \neq 0$.
Let $Ann_{\mathbb{Z}}(m) = l\mathbb{Z}$. If $l = 0$, then

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\text{any nonzero map}"] \arrow[d, hook, "a \mapsto am"'] & \mathbb{Q}/\mathbb{Z} \\ M \arrow[ur, dashed, "\exists f"'] & \end{tikzcd}$$

If $l \neq 0$, then

$$\begin{tikzcd} \mathbb{Z}/l\mathbb{Z} \arrow[r, "\frac{1}{l}"] \arrow[d, hook, "a \mapsto am"'] & \mathbb{Q}/\mathbb{Z} \\ M \arrow[ur, dashed, "\exists f"'] & \end{tikzcd}$$

□

Theorem 1.7.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories such that F is faithful ($\Leftrightarrow$ conservative : $F(A) = 0 \Rightarrow A = 0$).*

(1) *Suppose that $\mathcal{B}$ has enough injectives and that F has a right adjoint G : $\begin{tikzcd} \mathcal{A} \arrow[d, shift right, "F"'] \\ \mathcal{B} \arrow[u, shift right, "G"'] \end{tikzcd}$, then $\mathcal{A}$ has enough injectives. In cash : for every object $A \in \mathcal{A}$, choose a monomorphism $\alpha : F(A) \hookrightarrow I$ in $\mathcal{B}$ with $I \in Inj(\mathcal{B})$, then*

$$\begin{tikzcd} A \arrow[rr, hook, "G(\alpha)\circ\eta_A"] \arrow[dr, hook, "\eta_A"'] & & G(I) \\ & GF(A) \arrow[ur, hook, "G(\alpha)"'] & \end{tikzcd}$$

is a monomorphism into an injective object.

(2) *If $\mathcal{B}$ has enough projectives and F has a left adjoint E : $\begin{tikzcd} \mathcal{B} \arrow[d, shift right, "E"'] \\ \mathcal{A} \arrow[u, shift right, "F"'] \end{tikzcd}$, then $\mathcal{A}$ has enough projectives. For every $A \in \mathcal{A}$, choose an epimorphism $\beta : P \twoheadrightarrow F(A)$ with $P \in Proj(\mathcal{B})$, then $E(P) \xrightarrow{\epsilon_A \circ E(\beta)} A$ is an epimorphism from a projective object.*

Proof. (1) $\begin{tikzcd} \mathcal{A} \arrow[d, shift right, "F"'] \\ \mathcal{B} \arrow[u, shift right, "G"'] \end{tikzcd}$: G is left exact, thus it preserves monomorphisms. Under F, because $\epsilon_F \circ F(\eta) = id$, η preserves a (split) monomorphism. Since F is exact, $F(\ker \eta) = \ker(F(\eta)) = 0$. Since F is conservative, $\ker(\eta_A) = 0$ implies η_A is a monomorphism. Now we are left to prove the following, which is independently interesting. □

Proposition 1.7.10. *Consider an adjunction $\begin{tikzcd} \mathcal{A} \arrow[d, shift right, "F"'] \\ \mathcal{B} \arrow[u, shift right, "G"'] \end{tikzcd}$ between abelian categories.*

(1) *If F is exact, then G preserves injective objects.*
(2) *If G is exact, then F preserves projective objects.*

Proof. For $I \in Inj(\mathcal{B})$, the functor $\mathrm{Hom}_{\mathcal{A}}(-, G(I)) \xrightarrow[adj]{\sim} \mathrm{Hom}_{\mathcal{B}}(F(-), I) = \mathrm{Hom}_{\mathcal{B}}(-, I) \circ F$, which is a composition of exact functors, is exact. □

Corollary 1.7.11. *Let R be a ring, then R-Mod has enough injectives (and projectives, too).*

Proof. Consider $F : R\text{-}Mod \to Ab$ the forgetful functor (which is exact and conservative). Since Ab has enough injectives, we just need a right adjoint to F. We have

$$\begin{tikzcd} R\text{-}Mod \arrow[d, shift right, "F \simeq R\otimes_R -"'] \\ Ab \arrow[u, shift right, "\mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R, -)"'] \end{tikzcd}$$

by using ${}_{\mathbb{Z}}R_R \otimes_R {}_RM \cong_{\mathbb{Z}} M$ as an abelian group. □

Exercise 1.7.12. Unfold this corollary and the construction in Ab to explicitly describe $M \hookrightarrow I(M)$ for $M \in R\text{-}Mod$. [23]

Remark 1.7.13. When dealing with $Sh_{\mathcal{A}}(X)$ for a topological space X and an abelian category $\mathcal{A}$ (other than $\mathcal{A} = Ab$), one should require that $\mathcal{A}$ has all limits and (filtered) colimits, and that filtered colimits commute with products. This works for $\mathcal{A} = R\text{-}Mod$.

Corollary 1.7.14. *Let X be a topological space and $\mathcal{A}$ be a (nice) abelian category as above, e.g., $\mathcal{A} = Ab$ or $\mathcal{A} = R$-Mod. Then, $Sh_{\mathcal{A}}(X)$ has enough injectives.*

Proof. For every $x \in X$, consider $j_x : \{x\} \hookrightarrow X$ and $j_x^* : Sh_{\mathcal{A}}(X) \xrightarrow[\mathcal{F} \mapsto \mathcal{F}_x]{} Ab$. Then consider $F : Sh_{\mathcal{A}}(X) \xrightarrow[\mathcal{F} \mapsto (j_x^*\mathcal{F})_{x\in X} = (\mathcal{F}_x)_{x \in X}]{} \prod_{x\in X} \mathcal{A}$ where $\prod_{x \in X} \mathcal{A}$ is just componentwise. Then, $\prod_{x\in X}\mathcal{A}$ has enough injectives (componentwise). This functor F is exact and conservative. We just need a right adjoint. Let $(j_x)_* : \mathcal{A} \to Sh_{\mathcal{A}}(X)$ be defined by

$$((j_x)_* A)(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

for open $U \subseteq X$.

$$\begin{tikzcd} Sh_{\mathcal{A}}(X) \arrow[d, shift right, "(j_x)^*"'] \\ \mathcal{A} \arrow[u, shift right, "(j_x)_*"'] \end{tikzcd}$$

The counit $\epsilon_A : (A \cong)(j_x)^*(j_x)_*A \to A$ is the identity. The unit $\mathcal{F} \to (j_x)_*(j_x)^*\mathcal{F}$ is defined on every open $U \subseteq X$ by the obvious map

$$\mathcal{F}(U) = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

[23] We have $M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R, {}_{\mathbb{Z}}M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}R_R, \prod_{f \in \mathrm{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}) = \prod_f \mathrm{Hom}_Z({}_{\mathbb{Z}}R_R, \mathbb{Q}/\mathbb{Z})$.

Then, putting together,

$$\begin{tikzcd} Sh_{\mathcal{A}}(X) \arrow[d, shift right, "((j_x)^*)_{x\in X}"'] \\ \prod\limits_{x\in X} \mathcal{A} \arrow[u, shift right, "\prod_x (j_x)_*"'] \end{tikzcd}$$

□

Read more - Grothendieck: abelian categories, Tohoku J.

2.1. **Complexes.**

A basic idea of derived functors is that most homological complications would disappear if we were dealing only with projectives (or only with injectives).
Key example : A short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ with A_1 injective goes to an exact sequence under any additive functor F (e.g., left exact, but not right exact). [24]
Idea : To replace an object $A \in \mathcal{A}$ by injectives,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] & I_0 \arrow[rr] \arrow[dr, two heads] & & I_1 \arrow[rr] \arrow[dr, two heads] & & I_2 \arrow[r] & \cdots \\ & & & \bullet \arrow[ur, hook] & & \bullet \arrow[ur, hook] & & \end{tikzcd}$$

with all I_i injective. Really,

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & I_2 \arrow[r] & \cdots \end{tikzcd}$$

and this map is a quasi-isomorphism of complexes, i.e., an isomorphism in homology. Applying F to the second line yields:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow F(I_0) \longrightarrow F(I_1) \longrightarrow \cdots$$

which is the "complete" homological measure of A and its relation to F at least for F left exact. In particular, $H^0(F(I_\bullet)) \cong F(A)$ but the $H^i(F(I_\bullet))$ are also important. They are $R^iF(A)$, the right derived functors.

Definition 2.1.1. Let $\mathcal{A}$ be an additive category. A complex in $\mathcal{A}$ is a collection

$$\cdots \to A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \to \cdots$$

of objects $A_i \in \mathcal{A}$ and morphisms $d_i : A_i \to A_{i-1}$ such that $d_{i-1} \circ d_i = 0 (d^2 = 0)$ for all $i \in \mathbb{Z}$. (homological notation)
Alternatively, in cohomological notation,

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots$$

A morphism of complexes $f : (A_\bullet, d) \to (A'_\bullet, d')$ is a collection $f_i : A_i \to A'_i$ for all i such that $d'_i \circ f_i = f_{i-1} \circ d_i$. Let $Ch(\mathcal{A})$ be the category of complexes in $\mathcal{A}$ with morphisms of complexes.

Proposition 2.1.2. *(1) If $\mathcal{A}$ is additive, then $Ch(\mathcal{A})$ is additive.*
(2) If $\mathcal{A}$ is abelian, then $Ch(\mathcal{A})$ remains abelian.

Proof. Exercise! □

Remark 2.1.3. There is a fully faithful $\mathcal{A} \to Ch(\mathcal{A})$ defined by

$$A \longmapsto (\cdots \to 0 \to A \to 0 \to \cdots)$$

with A in degree 0, and this is exact if $\mathcal{A}$ is abelian.

[24] The image of a split exact sequence under an additive functor is split exact.

Definition 2.1.4. For $\mathcal{A}$ additive, we say that two morphisms $f, g : A_\bullet \to A'_\bullet$ in $Ch(\mathcal{A})$ are homotopic if there exists a homotopy $f \overset{\epsilon}{\sim} g$, that is, a collection of morphisms $\epsilon_i : A_i \to A'_{i+1}$ (NOT a morphism of complexes) such that $f = g + d'\epsilon + \epsilon d$ or explicitly, $f_i = g_i + d'_{i+1}\epsilon_i + \epsilon_{i-1}d_i$ for all $i \in \mathbb{Z}$. This notation is additive : $f \sim g \;\Leftrightarrow\; (f-g) \sim 0$.
Picture for $f \sim 0$:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r, "d"] \arrow[d] \arrow[dl] & A_i \arrow[r, "d"] \arrow[d, "f_i"'] \arrow[dl, "\epsilon_i"'] & A_{i-1} \arrow[r] \arrow[d] \arrow[dl, "\epsilon_{i-1}"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & A'_{i+1} \arrow[r, "d'"] & A'_i \arrow[r, "d'"] & A'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

where we have $f = d'\epsilon + \epsilon d$.

Remark 2.1.5. $\sim$ preserves $+$ and $\circ$: $f \sim f', g \sim g' \;\Rightarrow\; f \circ g \sim f' \circ g'$, etc. [25] Hence we get a well-defined homotopy category $K(\mathcal{A})$ of an additive category $\mathcal{A}$, with same abjects as $Ch(\mathcal{A})$ but morphisms up to homotopy:

$$\mathrm{Hom}_{K(\mathcal{A})}(A_\bullet, A'_\bullet) = \mathrm{Hom}_{Ch(\mathcal{A})}(A_\bullet, A'_\bullet)/\sim\; = \mathrm{Hom}_{Ch(\mathcal{A})}(A_\bullet, A'_\bullet)/(\text{subgroup of } f \sim 0)$$

Remark 2.1.6.
$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[rr, bend right, "\text{fully faithful}"'] & Ch(\mathcal{A}) \arrow[r, "\text{not faithful}"] & K(\mathcal{A})
\end{tikzcd}$$

Remark 2.1.7. If $\mathcal{A}$ is abelian, $K(\mathcal{A})$ is not, a priori!

Exercise 2.1.8 (Final Problem #3). Show that $K(\mathcal{A})$ is not abelian, in general. (Take $\mathcal{A} = Ab$ or $R\text{-}Mod$.) Find conditions under which $K(\mathcal{A})$ is abelian.

Remark 2.1.9. We will see later that $K(\mathcal{A})$ is actually triangulated (there are exact triangles

$$\begin{tikzcd}
& C \arrow[dl, "{[n]}"'] & \\
A \arrow[rr] & & B \arrow[ul]
\end{tikzcd}$$

which replace exact sequences) even if $\mathcal{A}$ is only additive : $A \to B \to C \to A[1]$.

Definition 2.1.10. A morphism $f : A_\bullet \to B_\bullet$ in $Ch(\mathcal{A})$ for additive $\mathcal{A}$ is called a homotopy equivalence if $[f] \in \mathrm{Hom}_{K(\mathcal{A})}(A_\bullet, B_\bullet)$ is an isomorphism: i.e., there exists $g : B_\bullet \to A_\bullet$ such that $f \circ g \sim id_{B_\bullet}$ and $g \circ f \sim id_{A_\bullet}$ in $Ch(\mathcal{A})$.

Remark 2.1.11. Any additive functor $F : \mathcal{A} \to \mathcal{B}$ between additive categories will induce $F = Ch(F) : Ch(\mathcal{A}) \to Ch(\mathcal{B})$ and $F = K(F) : K(\mathcal{A}) \to K(\mathcal{B})$. In particular, $F : Ch(\mathcal{A}) \to Ch(\mathcal{B})$ preserves homotopy equivalence.

Let's add the assumption that $\mathcal{A}$ is abelian.

Definition 2.1.12. Let $\mathcal{A}$ be abelian and $(A_\bullet, d) \in Ch(\mathcal{A})$ be a complex. For every $i \in \mathbb{Z}$, the i-th homology object $H_i(A_\bullet)$ is the coker$(\operatorname{im} d_{i+1} \hookrightarrow \ker d_i)$ where the morphism $\operatorname{im} d_{i+1} \to \ker d_i$ is the unique one such that

$$\begin{tikzcd}
A_{i+1} \arrow[rr, "d_{i+1}"] \arrow[dr, two heads] & & A_i \arrow[r, "d_i"] & A_{i-1} \\
& \operatorname{im} d_{i+1} \arrow[r, dashed, "\exists!"'] \arrow[ur, hook] & \ker d_i \arrow[u, hook] &
\end{tikzcd}$$

[25] We can show that, for example, $f \sim 0$ implies $h \circ f \sim 0$.

which exists because $d^2 = 0$.

Proposition 2.1.13. *For every $i \in \mathbb{Z}$, H_i defines a functor $H_i : Ch(\mathcal{A}) \to \mathcal{A}$. This functor is additive. Moreover, if $f \sim 0$, then $H_i(f) = 0$ for all $i \in \mathbb{Z}$. Hence we get a well-defined additive functor $H_i : K(\mathcal{A}) \to \mathcal{A}$.*

$$\begin{tikzcd}
A_{i+1} \arrow[rrr, "d^A_{i+1}"] \arrow[dr, two heads] \arrow[dd] & & & A_i \arrow[dd] & \\
& \operatorname{im} d^A_{i+1} \arrow[r, hook, dashed] \arrow[dd, red] \arrow[urr, hook] & \ker d^A_i \arrow[rr, two heads] \arrow[dd, red] \arrow[ur, hook] & & H_i(A_\bullet) \arrow[dd, red, dashed] \\
B_{i+1} \arrow[rrr, "d^B_{i+1}"] \arrow[dr, two heads] & & & B_i & \\
& \operatorname{im} d^B_{i+1} \arrow[r, hook, dashed] \arrow[urr, hook] & \ker d^B_i \arrow[rr, two heads] \arrow[ur, hook] & & H_i(B_\bullet)
\end{tikzcd}$$

Proof. Exercise! [26] □

Definition 2.1.14. We say that a morphism $f : A_\bullet \to B_\bullet$ (in $Ch(\mathcal{A})$ or $K(\mathcal{A})$) is a quasi-isomorphism if $H_i(f)$ is an isomorphism for all $i \in \mathbb{Z}$.

Corollary 2.1.15. *A homotopy equivalence is a quasi-isomorphism.*

Exercise 2.1.16. Let $A, B, C \in \mathcal{A}$ and $\alpha : A \to B$, $\beta : B \to C$. Consider

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "\alpha"] \arrow[d] & B \arrow[r] \arrow[d, "\beta"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

(1) When is this a morphism? [27]
(2) When is this a homotopy equivalence? [28]
(3) When is this a quasi-isomorphism? [29]
(4) Give (plenty of) examples of quasi-isomorphisms which are NOT homotopy equivalences.

[26] We have

$$\left(\ker d_i^A \to H_i(A_\bullet) \xrightarrow{H_i(f)} H_i(B_\bullet)\right) = \left(\ker d_i^A \to \ker d_i^B \to H_i(B)\right)$$
$$= \left(\ker d_i^A \to A_i \xrightarrow{\epsilon} B_{i+1} \to \operatorname{im} d_{i+1}^B \to \ker d_i^B \to H_i(B)\right) = 0$$

thus $H_i(f) = 0$.

[27] if and only if $\beta\alpha = 0$.

[28] if and only if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is split exact.

[29] if and only if $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is a short exact sequence.

Remark 2.1.17. We have the following.

$$
\begin{tikzcd}
A_{i+1} \arrow[rrr, "d_{i+1}"] \arrow[dr, two heads] & & & A_i \arrow[rrr, "d_i"] \arrow[dr, two heads, red] \arrow[drr, two heads, blue] & & & A_{i-1} \\
& \operatorname{im} d_{i+1} \arrow[r, hook, dashed] \arrow[urr, hook, red] & \ker d_i \arrow[ur, hook, blue] \arrow[dr, two heads] & & \operatorname{coker} d_{i+1} \arrow[r, dashed, two heads] & \operatorname{im} d_i \arrow[ur, hook] & \\
& & & H_i(A_\bullet) \arrow[ur, hook] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& 0 \arrow[r] \arrow[d] & H_i(A_\bullet) \arrow[d] \arrow[dddll, rounded corners, purple] \\
\operatorname{im} d_{i+1} \arrow[r, hook, red] \arrow[d, hook, dashed] & A_i \arrow[r, two heads, red] \arrow[d, equal] & \operatorname{coker} d_{i+1} \arrow[d, dashed, two heads, "\partial"', "\sim"'] \\
\ker d_i \arrow[r, hook, blue] \arrow[d, two heads] & A_i \arrow[r, two heads, blue] \arrow[d] & \operatorname{im} d_i \\
H_i(A_\bullet) \arrow[r] & 0 &
\end{tikzcd}
$$

One verifies that $H_i(A_\bullet)$ is simply the image of the unique map induced by $\ker d_i \hookrightarrow A_i \twoheadrightarrow \operatorname{coker} d_{i+1}$.

Lemma 2.1.18. *Let $A_\bullet$ be a complex in an abelian category $\mathcal{A}$.*

(1) We have

$$
\begin{aligned}
H_i(A_\bullet) &= \operatorname{coker}(\operatorname{im} d_{i+1} \to \ker d_i) \\
&= \ker(\operatorname{coker} d_{i+1} \to \operatorname{im} d_i) \\
&= \operatorname{im}(\ker d_i \to \operatorname{coker} d_{i+1})
\end{aligned}
$$

(2) There is a natural exact sequence:

$$0 \to H_i(A_\bullet) \to \operatorname{coker} d_{i+1} \xrightarrow{\tilde{\tilde{d_i}}} \ker d_{i-1} \to H_{i-1}(A_\bullet) \to 0$$

where $\tilde{\tilde{d_i}}$ is the unique map induced by d_i.

Proof. See above for (1). For (2), note that
$$
\begin{tikzcd}
\operatorname{coker} d_{i+1} \arrow[rr, "\tilde{\tilde{d_i}}"] \arrow[dr, two heads] & & \ker d_{i-1} \\
& \operatorname{im} d_i \arrow[ur, hook] &
\end{tikzcd}
$$
and use (1). $\square$

Theorem 2.1.19. *Let $\mathcal{A}$ be an abelian category and let*

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

be a short exact sequence in $Ch(\mathcal{A})$, i.e., $0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$ is a short exact sequence in $\mathcal{A}$ for all i. Then, there exists a natural long exact sequence:

$$\cdots \to H_i(A) \xrightarrow{H_i(f)} H_i(B) \xrightarrow{H_i(g)} H_i(C) \xrightarrow{\partial_i} H_{i-1}(A) \xrightarrow{H_{i-1}(f)} H_{i-1}(B) \to \cdots$$

(Think : $\partial_i = \partial_i(A_\bullet, B_\bullet, C_\bullet, f, g)$.)

Proof. Consider

$$\begin{tikzcd}
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, "d_i^A"] & B_i \arrow[r, "g_i"] \arrow[d, "d_i^B"] & C_i \arrow[r] \arrow[d, "d_i^C"] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"] & B_{i-1} \arrow[r, "g_{i-1}"] & C_{i-1} \arrow[r] & 0
\end{tikzcd}$$

By the (non-snake part of the) snake lemma, we get two exact sequences:

$$0 \to \ker d_i^A \xrightarrow{f} \ker d_i^B \xrightarrow{g} \ker d_i^C$$

$$\operatorname{coker} d_i^A \xrightarrow{f} \operatorname{coker} d_i^B \xrightarrow{g} \operatorname{coker} d_i^C \to 0$$

Hence we get a commutative diagram:

$$\begin{tikzcd}
 & H_i(A_\bullet) \arrow[r] \arrow[d] & H_i(B_\bullet) \arrow[r] \arrow[d] & H_i(C_\bullet) \arrow[d] \arrow[dddll, "\partial"] & \\
 & \operatorname{coker} d_{i+1}^A \arrow[r] \arrow[d, "\widetilde{d_i^A}"] & \operatorname{coker} d_{i+1}^B \arrow[r] \arrow[d, "\widetilde{d_i^B}"] & \operatorname{coker} d_{i+1}^C \arrow[r] \arrow[d, "\widetilde{d_i^C}"] & 0 \\
0 \arrow[r] & \ker d_{i-1}^A \arrow[r] \arrow[d] & \ker d_{i-1}^B \arrow[r] \arrow[d] & \ker d_{i-1}^C \arrow[d] & \\
 & H_{i-1}(A_\bullet) \arrow[r] & H_{i-1}(B_\bullet) \arrow[r] & H_{i-1}(C_\bullet) &
\end{tikzcd}$$

Use the previous lemma (2) with the snake lemma! □

2.2. **Projective and injective resolutions.**

Definition 2.2.1. Let $\mathcal{A}$ be an abelian category and $A \in \mathcal{A}$ be an object. An injective resolution of A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to \cdots$$

with all I^i injective in $\mathcal{A}$. In other words, it is a quasi-isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

A projective resolution of A is an exact sequence

$$\cdots \to P_1 \to P_0 \to A \to 0$$

with all P_i projective in $\mathcal{A}$, i.e., a quasi-isomorphism $P_\bullet \to c_0(A)$ with $P_\bullet \in Ch_{\geq 0}(Proj(\mathcal{A})) = Ch^{\leq 0}(Proj(\mathcal{A}))$.

Note 2.2.2. For any additive $\mathcal{A}$,

$$\begin{tikzcd} & Ch(\mathcal{A}) & \\ Ch^{\leq 0}(\mathcal{A}) = Ch_{\geq 0}(\mathcal{A}) \arrow[ur, hook, "\subseteq"] & & Ch_{\leq 0}(\mathcal{A}) = Ch^{\geq 0}(\mathcal{A}) \arrow[ul, hook', "\supseteq"'] \\ & \mathcal{A} \arrow[ul, hook', "\supseteq"] \arrow[ur, hook, "\subseteq"'] & \end{tikzcd}$$

and more generally,

$$Ch_{[a,b]}(\mathcal{A}) = \{X_\bullet \mid X_i = 0 \text{ except for } i \in [a,b]\}$$

$$Ch^{[a,b]}(\mathcal{A}) = \{X^\bullet \mid X^i = 0 \text{ except for } i \in [a,b]\} = Ch_{[-b,-a]}(\mathcal{A})$$

Proposition 2.2.3. *Let $\mathcal{A}$ be abelian.*

(1) If $\mathcal{A}$ has enough injectives, then any object has an injective resolution.
(2) If $\mathcal{A}$ has enougn projectives, then any object has a projective resolution.

Proof. (1) Let $A \in \mathcal{A}$. There exists a monomorphism $\xi_0 : A \hookrightarrow I^0 \in Inj(\mathcal{A})$. Consider $\operatorname{coker} \xi_0$. There exists a monomorphism $\xi_1 : \operatorname{coker} \xi_0 \hookrightarrow I^1 \in Inj(\mathcal{A})$. By induction, we construct exact sequences

$$\operatorname{coker}(\xi_i) \xhookrightarrow{\xi_{i+1}} I^{i+1} \twoheadrightarrow \operatorname{coker} \xi_{i+1}$$

for all $i \geq 0$. Putting those short exact sequences together, we get

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\xi_0"] & I^0 \arrow[rr, "d^0"] \arrow[dr, two heads] & & I^1 \arrow[r] & \cdots \arrow[r] & I^i \arrow[rr, "d^i"] \arrow[dr, two heads] & & I^{i+1} \arrow[r] & \cdots \\ & & & \operatorname{coker} \xi_0 \arrow[ur, hook, "\xi_1"'] & & & & \operatorname{coker} \xi_i \arrow[ur, hook, "\xi_{i+1}"'] & & \end{tikzcd}$$

in which the differentials $d^i : I^i \to I^{i+1}$ are defined as the compisotion $I^i \twoheadrightarrow \operatorname{coker} \xi_i \hookrightarrow I^{i+1}$.
(2) Dual. □

Proposition 2.2.4. *Let $\mathcal{A}$ be abelian.*

(1) Let $A, B \in \mathcal{A}$ and let $P_\bullet \xrightarrow{\xi_0} A$ be a projective resolution of A and $Q_\bullet \xrightarrow{\eta_0} B$ be a projective resolution of B. Let $f : A \to B$ be a morphism in $\mathcal{A}$. Then, there exists a morphism of complexes $f_\bullet : P_\bullet \to Q_\bullet$ such that $f \circ \xi_0 = \eta_0 \circ f_0$.

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r] \arrow[d, "f_n"] & \cdots \arrow[r] & P_0 \arrow[r, "\xi_0"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_0 \arrow[r, "\eta_0"] & B \arrow[r] & 0 \end{tikzcd}$$

Moreover, this $f_\bullet$ is unique up to homotopy, i.e., if $\widetilde{f}_\bullet : P_\bullet \to Q_\bullet$ is another morphism of complexes such that $f \circ \xi_0 = \eta_0 \circ \widetilde{f}_0$, then there exists $f \sim \widetilde{f}$.

(2) The dual : any morphism extends to injective resolutions in a unique way up to homotopy.

Proof. We have the following construction:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[rr, "d"] \arrow[dr, dashed, "(c)"] \arrow[dd, dashed, "\exists f_1", "(d)"'] & & P_0 \arrow[r, "\xi_0"] \arrow[dd, dashed, "\exists f_0", "(a)"'] & A \arrow[r] \arrow[dd, "f"] & 0 \\
& & \ker\eta_0 \arrow[dr, hook] \arrow[d, phantom, "(b)"] & & & \\
\cdots \arrow[r] & Q_1 \arrow[rr, "d'"'] \arrow[ur, two heads] & {} & Q_0 \arrow[r, two heads, "\eta_0"'] & B \arrow[r] & 0
\end{tikzcd}
$$

(a) P_0 is projective and $Q_0 \overset{\eta_0}{\twoheadrightarrow} B$
(b) $Q_\bullet \to B$ is exact
(c) $\eta_0 f_0 d = f \xi_0 d = 0$
(d) P_1 is projective and $Q_1 \twoheadrightarrow \ker\eta_0$

Suppose we have built $f_i : P_i \to Q_i$ for $i \leq n$ such that $d' f_i = f_{i-1} d$ for all i. Similarly, we get

$$
\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[rr] \arrow[dr, dashed] \arrow[dd, dashed, "\exists f_{n+1}"'] & & P_n \arrow[r] \arrow[dd, "f_n"] & P_{n-1} \arrow[r] \arrow[dd, "f_{n-1}"] & \cdots \\
& & \ker d' \arrow[dr, hook] & & & \\
\cdots \arrow[r] & Q_{n+1} \arrow[rr] \arrow[ur, two heads] & & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

For uniqueness, because the problem is additive, it suffices to show $f_\bullet \sim 0$ if $f = 0$. We have

$$
\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[rr, "d"] \arrow[dd, "f_1"'] & & P_0 \arrow[r, "\xi_0"] \arrow[dd, "f_0"] \arrow[dl, dashed, "(a)"] \arrow[ddll, dashed, bend right, "\exists \epsilon_0"', "(b)"] & A \arrow[r] \arrow[dd, "0"] & 0 \\
& & \ker\eta_0 \arrow[dr, hook] & & & \\
\cdots \arrow[r] & Q_1 \arrow[rr, "d'"'] \arrow[ur, two heads] & & Q_0 \arrow[r, "\eta_0"'] & B \arrow[r] & 0
\end{tikzcd}
$$

(a) $\eta_0 f_0 = 0$ and $Q_\bullet \to B$ is exact.
(b) $Q_1 \twoheadrightarrow \operatorname{im} d' = \ker\eta_0$ and P_0 is projective. So there exists $\epsilon_0 : P_0 \to Q_1$ such that $d'\epsilon_0 = f_0$.

Let's assume that we have constructed $\epsilon_i : P_i \to Q_{i+1}$ for all $i \leq n$ such that $f_i = d'\epsilon_i + \epsilon_{i-1} d$.

$$
\begin{tikzcd}
\cdots \arrow[r] & P_{n+2} \arrow[rr] \arrow[dd, "f_{n+2}"] & & P_{n+1} \arrow[r] \arrow[dd, "f_{n+1}"] \arrow[dl, dashed, "\alpha"] \arrow[ddll, dashed, bend right, "\epsilon_{n+1}"'] & P_n \arrow[r] \arrow[dd, "f_n"] \arrow[ddl, "\epsilon_n"] & P_{n-1} \arrow[r] \arrow[dd, "f_{n-1}"] \arrow[ddl, "\epsilon_{n-1}"] & \cdots \\
& & \ker d' \arrow[dr, hook] & & & & \\
\cdots \arrow[r] & Q_{n+2} \arrow[rr] \arrow[ur, two heads] & & Q_{n+1} \arrow[r] & Q_n \arrow[r] & Q_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

Consider $f_{n+1} - \epsilon_n d$ and apply d.

$$d(f_{n+1} - \epsilon_n d) = df_{n+1} - d\epsilon_n d = f_n d - d\epsilon_n d = (f_n - d\epsilon_n)d = \epsilon_{n-1} dd = 0$$

Then, there exists $\alpha : P_{n+1} \to \ker d'$ such that $f_{n+1} - \epsilon d = i\alpha$ where $i : \ker d' \hookrightarrow Q_{n+1}$. Since P_{n+1} is projective, there exists $\epsilon_{n+1} : P_{n+1} \to Q_{n+1}$. Then, $d'\epsilon_{n+1} = f_{n+1} - \epsilon_n d$ as needed. $\square$

Corollary 2.2.5. *Resolutions are unique up to unique (up to homotopy) homotopy equivalence.* [30]

Proof. Just apply the previous proposition to $A = B$ and $f = id$. □

Remark 2.2.6. The above means up to isomorphism of resolutions, i.e., not just $P_\bullet \xrightarrow{f} P'_\bullet$, but
$$\begin{tikzcd} P_\bullet \arrow[rr, "f"] \arrow[dr, "\xi_0"'] & & P'_\bullet \arrow[dl, "\xi'_0"] \\ & A & \end{tikzcd}$$
. In other words, resolutions = complexes of Inj/Proj with the map from/to A.

Recall that $K(-)$ is the homotopy category of any additive category where objects are complexes and morphisms are morphisms of complexes modulo homotopy equivalences. For instance, $K_{\geq 0}(Proj(\mathcal{A})) \subseteq K(\mathcal{A})$, $K^{\geq 0}(Inj(\mathcal{A})) = K_{\leq 0}(Inj(\mathcal{A}))$. We have
$$\begin{array}{rcll} c_0: & \mathcal{A} & \longrightarrow & K(\mathcal{A}) \\ & A & \longmapsto & (\cdots \to 0 \to \underbrace{A}_{0\text{th}} \to 0 \to \cdots) \end{array}$$

Consider $\mathcal{A} \overset{c_0}{\hookrightarrow} K_{\geq 0}(Proj(\mathcal{A})) \subseteq K(\mathcal{A})$.

Theorem 2.2.7. *Suppose $\mathcal{A}$ has enough projectives.*

(1) There exists a functor $\mathbb{P} : \mathcal{A} \to K_{\geq 0}(Proj(\mathcal{A})) \subseteq K(\mathcal{A})$ together with a natural transformation $\xi : \mathbb{P} \to c_0$
$$\begin{tikzcd} \mathbb{P}(A) \arrow[d, "\xi_A"'] & : & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\xi_0"] & 0 \arrow[r] \arrow[d] & \cdots \\ c_0(A) & : & \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$
such that $\xi_A : \mathbb{P}(A) \to c_0(A)$ is a quasi-isomorphism for all $A \in \mathcal{A}$.

(2) This pair $(\mathbb{P}, \xi)$ is unique up to unique isomorphism, i.e., if $(\mathbb{P}', \xi')$ is another such pair with $\mathbb{P}' : \mathcal{A} \to K_{\geq 0}(Proj(\mathcal{A}))$ with objectwise quasi-isomorphism and $\xi' : \mathbb{P}' \to c_0$, then there exists a unique isomorphism of such pairs, say $f : \mathbb{P} \to \mathbb{P}'$ (isomorphism of functors) such that $\xi' \circ f = \xi$.

Dually, if $\mathcal{A}$ has enough injectives, then there exists $\mathbb{I} : \mathcal{A} \to K^{\geq 0}(Inj(\mathcal{A}))$ with objectwise quasi-isomorphism $\eta : c_0 \to \mathbb{I}$ (as functors $\mathcal{A} \to K(\mathcal{A})$) which is unique up to unique isomorphism of such pairs.

Proof. Choose for every $A \in \mathcal{A}$ a projective resolution $P(A) := P_\bullet \xrightarrow{\xi_0} A$ (equivalently, choose a quasi-isomorphism $P(A) \xrightarrow{\xi_A} c_0(A)$.) Choose for every map $f : A \to B$ a lift
$$\begin{tikzcd} \mathbb{P}(A) \arrow[r, "\widehat{f}"] \arrow[d, "\xi_A"] & \mathbb{P}(B) \arrow[d, "\xi_B"] \\ c_0(A) \arrow[r, "c_0(f)"] & c_0(B) \end{tikzcd}$$
.

Set $\mathbb{P}(f) = [\widehat{f}] \in \mathrm{Hom}_{K(\mathcal{A})}(\mathbb{P}(A), \mathbb{P}(B))$. This yields the well-defined pair
$$(\mathbb{P} : \mathcal{A} \to K_{\geq 0}(Proj(\mathcal{A})), \xi : \mathbb{P} \to c_0)$$
as in (1). We have $\mathbb{P}(f \circ g) = \mathbb{P}(f) \circ \mathbb{P}(g)$ by the following argument. Choose lifts $\widehat{f}, \widehat{g}$ so that $\mathbb{P}(f) = [\widehat{f}], \mathbb{P}(g) = [\widehat{g}]$. Then observe that $\widehat{f} \circ \widehat{g}$ is a lift of $f \circ g$. By the previous proposition (uniqueness of lift), $\widehat{f \circ g} \sim \widehat{f} \circ \widehat{g}$. Hence, $\mathbb{P}(f) \circ \mathbb{P}(g) = [\widehat{f}] \circ [\widehat{g}] = [\widehat{f \circ g}] = \mathbb{P}(f \circ g)$. For (2),

[30] thus give the same homology/cohomology.

same story : at each $A \in \mathcal{A}$, consider $\xi_A : \mathbb{P} \to c_0(A)$ and $\xi'_A : \mathbb{P}' \to c_0(A)$. By existence and uniqueness of lift, we have

$$\begin{tikzcd}
\mathbb{P}(A) \arrow[rr, "\exists!", "f_A"'] \arrow[dr, "\xi_A"'] & & \mathbb{P}'(A) \arrow[dl, "\xi'_A"] \\
& c_0(A) &
\end{tikzcd}$$

Check the rest as an exercise. $\square$

Definition 2.2.8. If $\mathcal{A}$ has enough projectives, the (unique) functor $\mathbb{P} : \mathcal{A} \to K_{\geq 0}(Proj(\mathcal{A}))$ in the unique pair $(\mathbb{P}, \xi : \mathbb{P} \to c_0)$ is the projective resolution functor. Dually, if $\mathcal{A}$ has enough injectives, there is the injective resolution functor $\mathbb{I} : \mathcal{A} \to K^{\geq 0}(Inj(\mathcal{A}))$ uniquely characterized by the existence of a natrual quasi-isomorphism $c_0(A) \to \mathbb{I}(A)$ for $A \in \mathcal{A}$.

Remark 2.2.9. For a functor $F : \mathcal{A} \to \mathcal{B}$, we can consider various compositions of the following functors:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "c_0"] \arrow[dr, "\mathbb{P}"'] & K(\mathcal{A}) \arrow[r, "K(F)"] & K(\mathcal{B}) \arrow[r, "H_i"] & \mathcal{B} \\
& K_{\geq 0}(Proj(\mathcal{A})) \arrow[u, hook] & &
\end{tikzcd}$$

Note that the triangle on the left is NOT commutative.

Theorem 2.2.10 (Horseshoe Lemma). *Let $0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$. Let $P'_\bullet \xrightarrow{\xi'_0} A'$ and $P''_\bullet \xrightarrow{\xi''_0} A''$ be projective resolutions. Then, there exists a projective resolution $P_\bullet \xrightarrow{\xi_0} A$ and lifts*

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r, "\widehat{\alpha}'"] \arrow[d] & P_\bullet \arrow[r, "\widehat{\alpha}''"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] & A \arrow[r, "\alpha''"] & A'' \arrow[r] & 0
\end{tikzcd}$$

such that the sequence of complexes $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ is exact in $Ch(\mathcal{A})$, i.e., degree-wise exact. Hence, in particular, $P_i \cong P'_i \oplus P''_i$ for all i.

Proof. Let $P_0 = P'_0 \oplus P''_0$. Since P''_0 is projective and α'' is an epimorphism, we have $\xi_0 : P'_0 \oplus P''_0 \to A$ which makes the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r] \arrow[d, "\xi'_0"] & P'_0 \oplus P''_0 \arrow[r] \arrow[d, dashed, "\xi_0"] & P''_0 \arrow[r] \arrow[d, "\xi''_0"] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "\alpha'"] & A \arrow[r, two heads, "\alpha''"] & A'' \arrow[r] & 0
\end{tikzcd}$$

Note that, by snake, ξ_0 is epic. Then we have

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker \xi_0' \arrow[r] \arrow[d] & \ker \xi_0 \arrow[r] \arrow[d] & \ker \xi_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Apply the same to the following:

$$\begin{tikzcd}
 & P_1' \arrow[d, two heads] & & P_1'' \arrow[d, two heads] & \\
0 \arrow[r] & \ker \xi_0' \arrow[r, hook] & \ker \xi_0 \arrow[r, two heads] & \ker \xi_0'' \arrow[r] & 0
\end{tikzcd}$$

Hence the result by induction. □

Lemma 2.2.11 (Schanuel). *Suppose $A \in \mathcal{A}$ for an abelian category $\mathcal{A}$ and*

$$0 \to B \to P_n \to \cdots \to P_0 \to A \to 0$$

$$0 \to C \to Q_n \to \cdots \to Q_0 \to A \to 0$$

be exact sequences with all P_i, Q_j projective. (Note that we have same n.) Then, there are projective objects P, Q such that $B \oplus P \simeq C \oplus Q$. More precisely,

$$B \oplus Q_n \oplus P_{n-1} \oplus \cdots \oplus (P_0 \text{ or } Q_0) \simeq C \oplus P_n \oplus Q_{n-1} \oplus \cdots \oplus (Q_0 \text{ or } P_0)$$

Proof. When $n = 0$, we have the following:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "(b)"] & P \arrow[r, two heads] \arrow[u] \arrow[dr, phantom, "(a)"] & A \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & B \arrow[r] & D \arrow[r] \arrow[u] \arrow[dr, phantom, "(b)"] & Q \arrow[r] \arrow[u, two heads] & 0 \\
 & & C \arrow[r, equal] \arrow[u] & C \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

(a) pull-back
(b) general property of pull-back along epimorphisms (see Lemma 1.2.5 and Corollary 1.4.11)

Since P and Q are projective, the middle sequence split:

$$C \oplus P \simeq D \simeq B \oplus Q$$

For $n \geq 1$, the sequence

$$0 \to B \to P_n \to \cdots \to P_2 \to P_1 \oplus Q_0 \to A' \oplus Q_0 \to 0$$

exact for $A' \hookrightarrow P_0 \twoheadrightarrow A$. Similarly,

$$0 \to C \to Q_n \to \cdots \to Q_2 \to Q_1 \oplus P_0 \to A'' \oplus P_0 \to 0$$

is exact for $A'' \hookrightarrow Q_0 \twoheadrightarrow A$. We have $A' \oplus Q_0 \simeq A'' \oplus P_0$ by $n = 0$ case. By induction, we have the result. $\square$

2.3. **Left and right derived functors.**

Definition 2.3.1. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

(1) Suppose that $\mathcal{A}$ has enough projectives. Then the i-th left derived functor of F for $i \geq 0$ is the following composition:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathbb{P}"] \arrow[rrr, bend right, "L_iF"'] & K_{\geq 0}(Proj(\mathcal{A})) \arrow[r, "F=K(F)"] & K(\mathcal{B}) \arrow[r, "H_i"] & \mathcal{B} \end{tikzcd}$$

In cash, $L_iF(-) = H_i(F(\mathbb{P}(-)))$.

(2) Suppose that $\mathcal{A}$ has enough injectives. Then the i-th right derived functor of F is the composition:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\mathbb{I}"] \arrow[rrr, bend right, "R^iF"'] & K^{\geq 0}(Inj(\mathcal{A})) \arrow[r, "F=K(F)"] & K(\mathcal{B}) \arrow[r, "H^i"] & \mathcal{B} \end{tikzcd}$$

<u>**Hypothesis**</u> : For this section, $\mathcal{A}$ is assumed to have enough injectives (resp. projectives) as needed.

Proposition 2.3.2. *Let $A \in \mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Let $P_\bullet \xrightarrow{\xi_0} A$ be some projective resolution. Then there exists a canonical isomorphism $L_iF(A) \xrightarrow{\sim} H_i(F(P_\bullet))$. Moreover, for every morphism $f : A \to B$ in $\mathcal{A}$ and any choice of $Q_\bullet \xrightarrow{\eta_0} B$ of a projective resolution and any choice of a lift $f_\bullet : P_\bullet \to Q_\bullet$ of f, the following square commutes in $\mathcal{B}$:*

$$\begin{tikzcd} L_iF(A) \arrow[r, "\sim"] \arrow[d, "L_iF(f)"'] & H_i(F(P_\bullet)) \arrow[d, "H_i(F(f_\bullet))"] \\ L_iF(B) \arrow[r, "\sim"] & H_i(F(Q_\bullet)) \end{tikzcd}$$

Dually, the same holds for injective resolutions and right derived functors.

Proof. The projective resolution $\mathbb{P}(A)$ is unique up to unique isomorphism, as an object in $K_{\geq 0}(Proj(\mathcal{A}))$ together with the map $\mathbb{P}(A) \to A$. The same for the maps (the obvious square commutes in $K_{\geq 0}(Proj(\mathcal{A}))$:

$$\begin{tikzcd} \mathbb{P}(A) \arrow[r, "\sim"] \arrow[d, "\mathbb{P}(f)"'] & P_\bullet \arrow[d, "f_\bullet"] \\ \mathbb{P}(B) \arrow[r, "\sim"] & Q_\bullet \end{tikzcd}$$

since

$$\begin{tikzcd} \mathbb{P}(A) \arrow[r, "\xi"] \arrow[d, shift left] \arrow[d, shift right] & A \arrow[d, "f"] \\ Q_\bullet \arrow[r, "\eta_0"] & B \end{tikzcd}$$

are two lifts, thus same in $K_{\geq 0}$.) Then, apply the functor $K_{\geq 0}(Proj(\mathcal{A})) \xrightarrow{F} K(\mathcal{B}) \xrightarrow{H_i} \mathcal{B}$. $\square$

Exercise 2.3.3. Show that for $F : \mathcal{A} \to \mathcal{B}$ additive between abelian categories, the induced $K(\mathcal{A}) \to K(\mathcal{B})$ preserves quasi-isomorphisms if and only if F is exact. [31]

Theorem 2.3.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be additive. Suppose $\mathcal{A}$ has enough projectives (resp. injectives) and let $0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$ be a short exact sequence in $\mathcal{A}$. Then, there exists a natural canonical long exact sequence:*

$$\cdots \to L_1F(A'') \xrightarrow{\partial} L_0(A') \xrightarrow{L_0F(\alpha')=\alpha'_*} L_0F(A) \xrightarrow{L_0F(\alpha'')=\alpha''_*} L_0F(A'') \to 0$$

(resp. $\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\partial} R^{i+1}F(A') \to \cdots$.) If moreover F is right exact (resp. left exact), then $L_0 \simeq F$ (resp. $R^0 \simeq F$.)

Proof. By the Horseshoe lemma, we can find projective resolutions:

$$\begin{tikzcd} 0 \arrow[r] & P'_\bullet \arrow[r] \arrow[d] & P_\bullet \arrow[r] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

degree-wise (split) exact. Since F is additive, $0 \to F(P'_\bullet) \to F(P_\bullet) \to F(P''_\bullet) \to 0$ is degree-wise (split) exact. This lives in $Ch(\mathcal{B})$. Then apply the homology long exact sequence (in $\mathcal{B}$). If F is right exact and

$$\cdots \to P_1 \to P_0 \to A \to 0$$

is a projective resolution, then this gives

$$L_0(A) = H_0(\cdots \to F(P_1) \to F(P_0) \to 0 \to \cdots) = F(A)$$

in $\mathcal{B}$. $\square$

Definition 2.3.5. Let $F : \mathcal{A} \to \mathcal{B}$ be a (right exact) additive functor between abelian categories. An object $E \in \mathcal{A}$ is called (left) F-acyclic if $L_iF(E) = 0$ for all $i > 0$.

Example 2.3.6. Projective objects of $\mathcal{A}$ are left acyclic. ($0 \to P \xrightarrow{1} P \to 0$ is a projective resolution.)

Lemma 2.3.7. *Let $F : \mathcal{A} \to \mathcal{B}$ be right exact.*

(1) If $A' \hookrightarrow A \twoheadrightarrow E$ is a short exact sequence in $\mathcal{A}$ and E is F-acyclic, then $F(A') \hookrightarrow F(A) \twoheadrightarrow F(E)$ is a short exact sequence in $\mathcal{B}$.
(2) If $A \hookrightarrow E \twoheadrightarrow E'$ is a short exact sequence in $\mathcal{A}$ and E, E' are F-acyclic, then A is F-acyclic.
(3) If $E_\bullet \in Ch_+(\mathcal{A})$ is a homologically bounded below complex of F-acyclic which is exact, then $F(E_\bullet) \in Ch_+(\mathcal{B})$ is exact.
(4) If $f_\bullet : E_\bullet \to E'_\bullet$ is a quasi-isomorphism of (homologically) bounded below complexes of F-acyclics, then $F(f_\bullet)$ is a quasi-isomorphism.

[31] see 2.1.16

Proof. (1) We have $0 = L_1F(E) \to F(A') \to F(A) \to F(E) \to 0$.
(2) For all $i \geq 1$, $0 = L_{i+1}F(E') \to L_iF(A) \to L_iF(E) = 0$ is exact.
(3) Consider the following

$$\begin{tikzcd}
\cdots \arrow[r] & E_{m+2} \arrow[rr, "d"] \arrow[dr, two heads] & & E_{m+1} \arrow[rr, "d"] \arrow[dr, two heads] & & E_m \arrow[r] & 0 \arrow[r] & \cdots \\
& & A_{m+1} \arrow[ur, hook] & & A_m \arrow[ur, hook] & & &
\end{tikzcd}$$

By induction on (2), all $A_i = \operatorname{im} d_{i+1}$ are F-acyclic because (by exactness of $E_\bullet$) $A_{m+1} \hookrightarrow E_{m+1} \twoheadrightarrow A_m$ is a short exact sequence in $\mathcal{A}$. Thus by (1), $F(A_{m+1}) \hookrightarrow F(E_{m+1}) \twoheadrightarrow F(A_m)$ is exact. Thus

$$\begin{tikzcd}
\cdots \arrow[r] & F(E_{m+2}) \arrow[rr, "F(d)"] \arrow[dr, two heads] & & F(E_{m+1}) \arrow[rr, "F(d)"] \arrow[dr, two heads] & & F(E_m) \arrow[r] & 0 \arrow[r] & \cdots \\
& & F(A_{m+1}) \arrow[ur, hook] & & F(A_m) \arrow[ur, hook] & & &
\end{tikzcd}$$

is exact, hence (3).
(4) Let $f_\bullet : E_\bullet \to E'_\bullet$ be a quasi-isomorphism. We first reduce to the case where $f_i : E_i \to E'_i$ is an epimorphism in each degree. It is enough to add to $E_\bullet$ a complex of F-acyclic $\widehat{E}_\bullet$ which is homotopic to 0. Take $\widehat{E}_\bullet$ to be the $\oplus$ of complexes of the form $(\cdots 0 \to E'_i \xrightarrow{1} E'_i \to 0 \cdots)$, i.e.,

$$\widehat{E}_\bullet = \bigoplus_{i \in \mathbb{Z}} (\cdots \to 0 \to E'_i \to E'_i \to 0 \to \cdots)$$

then we have

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & E'_i \arrow[r, "id"] \arrow[d, "id"] & E'_i \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & E'_{i+1} \arrow[r] & E'_i \arrow[r] & E'_{i-1} \arrow[r] & E'_{i-2} \arrow[r] & \cdots
\end{tikzcd}$$

Thus this defines $\widehat{E}_\bullet \xrightarrow{\widehat{f}} E'_\bullet$ degree-wise epimorphism $\widehat{f} \sim 0$. [32] Then contemplate $E \oplus \widehat{E} \xrightarrow{(f \;\; \widehat{f})} E'$. Since $F(\widehat{f}_\bullet) \sim 0$, we are reduced to the special case where $f_\bullet : E_\bullet \to E'_\bullet$ is a bounded below F-acyclic quasi-isomorphism and each f_i is an epimorphism. We want to show that $F(f_\bullet)$ is a quasi-isomorphism. Consider $A_\bullet = \ker f_\bullet$ in $Ch(\mathcal{A})$. We have an exact sequence $A_\bullet \hookrightarrow E_\bullet \overset{f_\bullet}{\twoheadrightarrow} E'_\bullet$. By (2) and the short exact sequence $A_i \hookrightarrow E_i \overset{f_i}{\twoheadrightarrow} E'_i$, we see that A_i is F-acyclic. By the long exact sequence in $H_\bullet$

$$\to H_i(E) \xrightarrow{\sim} H_i(E') \to H_{i-1}(A) \to H_{i-1}(E) \xrightarrow{\sim} H_{i-1}(E')$$

($\mathcal{A}$ abelian), $H_{i-1}(A_\bullet) = 0$. So $A_\bullet$ is a bounded below exact complex of F-acyclic, thus by (3), $F(A_\bullet)$ is exact. Since $A_i \hookrightarrow E_i \twoheadrightarrow E'_i$ and by (1), $F(A_\bullet) \to F(E_\bullet) \to F(E'_\bullet)$ is degree-wise exact. By

[32] $\widehat{f}$ is defined as follows.

$$\begin{tikzcd}
\cdots \arrow[r] & E'_i \oplus E'_{i+1} \arrow[r, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"] \arrow[d, "(id \;\; d_{i+1})"'] & E'_{i-1} \oplus E'_i \arrow[r, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}"] \arrow[d, "(id \;\; d_i)"] \arrow[dl, "\epsilon_i"'] & E'_{i-2} \oplus E'_{i-1} \arrow[r] \arrow[d, "(id \;\; d_{i-1})"] & \cdots \\
\cdots \arrow[r] & E'_i \arrow[r, "d_i"] & E'_{i-1} \arrow[r, "d_{i-1}"] & E'_{i-2} \arrow[r] & \cdots
\end{tikzcd}$$

Here the maps $\epsilon_i : E'_{i-1} \oplus E'_i \xrightarrow{(0 \;\; 1)} E'_i$ gives $\widehat{f} = (id \;\; d_\bullet) \sim 0$.

long exact sequence in $H_\bullet$ in $\mathcal{B}$

$$\to 0 = H_i(F(A_i)) \to H_i(F(E_i)) \to H_i(F(E_i')) \to 0$$

($\mathcal{B}$ abelian), $H_\bullet(F(f_\bullet))$ is an isomphism, i.e., $F(f_\bullet)$ is a quasi-isomorphism. $\square$

Exercise 2.3.8 (Final Problem #4). If $B \hookrightarrow E \twoheadrightarrow A$ is exact in $\mathcal{A}$ and E is $F-$acyclic, then there exists a natural isomorphism $L_{i+1}F(A) \simeq L_iF(B)$ for $i \geq 1$. More generally, if

$$0 \to B \to E_m \to \cdots \to E_1 \to A \to 0$$

is exact and all E_i are $F-$acyclic, then $L_{i+m}F(A) \simeq L_iF(B)$ for all $i \geq 1$.

Theorem 2.3.9 (Derived functors using acyclic objects)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian and $F : \mathcal{A} \to \mathcal{B}$ be right exact. Suppose that $\mathcal{A}$ has enough projectives. Let $A \in \mathcal{A}$. For any resolution of A by F-acyclics*

$$\cdots \to E_n \to \cdots \to E_1 \to E_0 \to A \to 0,$$

there exists a natural and canonical isomorphism $L_iF(A) \simeq H_iF(E_\bullet)$ for all $i \geq 0$. Dually for right derived functors.

Proof. Let $P_\bullet \xrightarrow{\xi_0} A$ be a projective resolution. We know that there exists a (unique up to homotopy) morphism $f_\bullet : P_\bullet \to E_\bullet$ such that

$$\begin{array}{ccccccc}
\cdots \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow 0 \\
 & \downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow & \\
\cdots \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & A & \longrightarrow 0
\end{array}$$

So $H_i(f_\bullet) : H_i(P_\bullet) \to H_i(E_\bullet)$ is an isomorphism for all $i \in \mathbb{Z}_{\geq 0}$. So $f_\bullet$ is a quasi-isomorphism of bounded below complexes of F-acyclic (because projectives are). By the lemma, $F(f_\bullet)$ remains a quasi-isomorphism. Hence $H_i(F(f_\bullet)) : L_iF(A) = H_iF(P_\bullet) \xrightarrow{\sim} H_iF(E_\bullet)$. $\square$

Remark 2.3.10. If $\mathcal{A}$ doesn't have enough projectives, but has enough objects in a nice subcategory $\mathcal{E} \subseteq \mathcal{A}$, then we can define L_iF by the formula of the theorem. "Nice" means

(1) If $A \hookrightarrow E \twoheadrightarrow E''$ with $E, E' \in \mathcal{E}$, then $A \in \mathcal{E}$.
(2) If $A' \hookrightarrow A \twoheadrightarrow E$ with $E \in \mathcal{E}$ (is it enough all in $\mathcal{E}$?) then $FA' \hookrightarrow FA \twoheadrightarrow FE$ is exact.

2.4. Ext and Tor.

We want to derive Hom and $\otimes$. Let us discuss the situation of a functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are abelian and F is additive in each variable : $F(-, B) : \mathcal{A} \to \mathcal{C}$ and $F(A, -) : \mathcal{B} \to \mathcal{C}$ are additive for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Double complexes : Let $\mathcal{C}$ be additive. We can consider objects in $ChCh(\mathcal{C})$ as double complexes

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
\cdots \longrightarrow & C_{ij} & \xrightarrow{d^h_{ij}} & C_{i-1,j} & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle d^v_{ij}} & & \downarrow & \\
\cdots \longrightarrow & C_{i,j-1} & \longrightarrow & C_{i-1,j-1} & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & \\
 & \vdots & & \vdots &
\end{array}$$

i.e., the data of objects $C_{ij} \in \mathcal{C}$ for $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ and morphisms $d^v_{ij} : C_{ij} \to C_{i,j-1}$ and $d^h_{ij} : C_{ij} \to C_{i-1,j}$ such that $d^v d^v = 0$, $d^h d^h = 0$ and $d^h d^v = d^v d^h$.
Suppose that $C_{\bullet\bullet}$ is bounded below in both directions : there exist m, n such that $C_{ij} = 0$ if $i < m$ or $j < n$. Then we define $Tot(C_{\bullet\bullet})$ to be the complex by

$$\begin{tikzcd}
Tot(C_{\bullet\bullet})_k = \bigoplus\limits_{i+j=k} C_{ij} \arrow[r, "d"] & \bigoplus\limits_{i'+j'=k-1} C_{i'j'} = Tot(C_{\bullet\bullet})_{k-1} \\
C_{ij} \arrow[u, hook] \arrow[r, "\begin{pmatrix} d^h_{ij} \\ (-1)^i d^v_{ij} \end{pmatrix}"] & C_{i-1,j} \oplus C_{i,j-1} \arrow[u, hook]
\end{tikzcd}$$

Check this is a complex! [33]

Remark 2.4.1. If you need to handle unbounded double complexes, there is a choice between $Tot^{\coprod}$ and $Tot^{\prod}$ to replace the above $\bigoplus\limits_{\text{finite}}$.

Example 2.4.2. For $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, we get

$$\begin{tikzcd}
Ch_+(\mathcal{A}) \times Ch_+(\mathcal{B}) \arrow[rr, "F = F^{tot}"] \arrow[dr, "F"'] & & Ch_+(\mathcal{C}) \\
& Ch_+ Ch_+(\mathcal{C}) \arrow[ur, "Tot^{\oplus}"'] &
\end{tikzcd}$$

which is defined by $F^{tot}(A_\bullet, B_\bullet) = Tot^{\oplus} F(A_\bullet, B_\bullet)$.

Exercise 2.4.3. $F^{tot}(-,-) : Ch_+(\mathcal{A}) \times Ch_+(\mathcal{B}) \to Ch_+(\mathcal{C})$ preserves homotopy equivalence and degree-wise split short exact sequences in each variable : if $A'_\bullet \hookrightarrow A_\bullet \twoheadrightarrow A''_\bullet$ is a degree-wise split exact sequence in $Ch_+(\mathcal{A})$ and $B_\bullet \in Ch_+(\mathcal{B})$ is arbitrary, then

$$F^{tot}(A'_\bullet, B_\bullet) \to F^{tot}(A_\bullet, B_\bullet) \to F^{tot}(A''_\bullet, B_\bullet)$$

is a degree-wise split exact sequence in $Ch_+(\mathcal{C})$. This is purely additive. [34]

Theorem 2.4.4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian and $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be additive in each variable. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives and that F is right exact (meaning that $F(-,B) : \mathcal{A} \to \mathcal{C}$ is right exact for all $B \in \mathcal{B}$ and $F(A,-) : \mathcal{B} \to \mathcal{C}$ is right exact for all $A \in \mathcal{A}$.) Suppose*

(1) $F(P,-) : \mathcal{B} \to \mathcal{C}$ is exact if $P \in \mathcal{A}$ is projective.
(2) $F(-,Q) : \mathcal{A} \to \mathcal{C}$ is exact if $Q \in \mathcal{B}$ is projective.

Then, there exist natural and canonical isomorphisms $(L_i F(A,-))(B) \cong (L_i F(-,B))(A)$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. In cash, it means that if $P_\bullet \to A$ and $Q_\bullet \to B$ are projective resolutions, then $H_i(F(A, Q_\bullet)) \cong H_i(F(P_\bullet, B))$.

We need the following.

Lemma 2.4.5. *Let $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be as in the theorem. Let $f_\bullet : A_\bullet \to A'_\bullet$ be a quasi-isomorphism of bounded below complex in $\mathcal{A}$. Let $Q_\bullet$ be a bounded below complex of projectives in $\mathcal{B}$. Then, $F^{tot}(f_\bullet, id_{Q_\bullet}) : F^{tot}(A_\bullet, Q_\bullet) \to F^{tot}(A'_\bullet, Q_\bullet)$ is a quasi-isomorphism.*

[33] Basically from the matrix representation of $d : Tot(\mathcal{C}_{\bullet\bullet})_k \to Tot(\mathcal{C}_{\bullet\bullet})_{k-1}$. In the product of matrices $d \circ d$, we have $d^v d^v = 0$, $d^h d^h = 0$ and $(\pm d^v \;\; d^h)\begin{pmatrix} d^h \\ \mp d^v \end{pmatrix} = 0$.

[34] An additive functor preserves split exact sequences. Here $F(-, B_\bullet) : Ch_+(\mathcal{A}) \to Ch_+ Ch_+(\mathcal{C})$ and $Tot^{\oplus} : Ch_+ Ch_+(\mathcal{C}) \to Ch_+(\mathcal{C})$ are both additive.

Proof. Consider the following.

$$\begin{array}{ccc} F^{tot}(A_\bullet, Q_\bullet)_n & & F^{tot}(A'_\bullet, Q_\bullet)_n \\ \| & & \| \\ \bigoplus_{i+j=n} F(A_i, Q_j) & \xrightarrow{F^{tot}(f_\bullet, id)} & \bigoplus_{i+j=n} F(A'_i, Q_j) \\ \Big\downarrow{\scriptstyle d=F(d,id)+(-1)^i F(id,d)} & & \Big\downarrow \\ F^{tot}(A_\bullet, Q_\bullet)_{n-1} & & F^{tot}(A'_\bullet, Q_\bullet)_{n-1} \end{array}$$

Since, in each degree n, only finitely many Q_j intervene, we can assume that Q is actually bounded on both sides : $Q_j = 0$ unless $p \le j \le q$. Proceed by induction on $q - p$.
For $q - p = 0$, we have $Q_\bullet = (\cdots \to 0 \to Q_p \to 0 \to \cdots)$. Then, $F^{tot}(-, Q_\bullet) = F(-, Q_p)$ somewhat shifted in degree. So, it suffices to show that $F(-, Q)$ preserves quasi-isomorphism for $Q \in Proj(\mathcal{B})$. This follows from (2).
Suppose the result for $q - p = r$ and contemplate $Q_\bullet$ with $Q_j = 0$ except $p \le j \le q$ with $q - p = r + 1$. We have a degree-wise split short exact sequence

$$\begin{array}{cccccccccccccccc} Q'_\bullet & : & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Q_{q-1} & \longrightarrow & \cdots & \longrightarrow & Q_p & \longrightarrow & 0 & \longrightarrow \cdots \\ \Big\downarrow{\scriptstyle \beta} & & & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow & \\ Q_\bullet & : & \cdots & \longrightarrow & 0 & \longrightarrow & Q_q & \longrightarrow & Q_{q-1} & \longrightarrow & \cdots & \longrightarrow & Q_p & \longrightarrow & 0 & \longrightarrow \cdots \\ \Big\downarrow{\scriptstyle \beta'} & & & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow & \\ Q''_\bullet & : & \cdots & \longrightarrow & 0 & \longrightarrow & Q_q & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

By the additive comments before the theorem, $F^{tot}(A_\bullet, -)$ and $F^{tot}(A'_\bullet, -)$ will preserve (degree-wise split) exactness of such sequences. So the rows below are short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{tot}(A_\bullet, Q'_\bullet) & \xrightarrow{F^{tot}(id_{A_\bullet}, \beta)} & F^{tot}(A_\bullet, Q_\bullet) & \xrightarrow{F^{tot}(id_{A_\bullet}, \beta')} & F^{tot}(A_\bullet, Q''_\bullet) & \longrightarrow & 0 \\ & & {\scriptstyle (a)}\Big\downarrow{\scriptstyle F^{tot}(f_\bullet, id_{Q'_\bullet})} & & \Big\downarrow & & \Big\downarrow{\scriptstyle (a)} & & \\ 0 & \longrightarrow & F^{tot}(A'_\bullet, Q'_\bullet) & \longrightarrow & F^{tot}(A'_\bullet, Q_\bullet) & \longrightarrow & F^{tot}(A'_\bullet, Q''_\bullet) & \longrightarrow & 0 \end{array}$$

These are complexes in $\mathcal{C}$. Apply the $H_\bullet$ long exact sequence in $\mathcal{C}$ [35] , then the two vertical maps $H_\bullet(F^{tot}(f_\bullet, id_{Q'_\bullet}))$ and $H_\bullet(F^{tot}(f_\bullet, id_{Q''_\bullet}))$ (induced by (a)) are quasi-isomorphisms by induction on the length of Q-complexes. By 5-lemma in $\mathcal{C}$, the map $H_\bullet(F^{tot}(f_\bullet, id_{Q_\bullet}))$ is an isomorphism. So $F^{tot}(f_\bullet, id_{Q_\bullet})$ is a quasi-isomorphism. $\square$

Proof of Theorem 2.4.4. Consider $P_\bullet \xrightarrow{\xi} c_0(A)$ and $Q_\bullet \xrightarrow{\eta} c_0(B)$ quasi-isomorphisms with $P_\bullet \in Ch_+(Proj(\mathcal{A})), Q_\bullet \in Ch_+(Proj(\mathcal{B}))$. Consider $F^{tot}(-,-)$ on these :

$$\begin{array}{ccc} F^{tot}(P_\bullet, Q_\bullet) & \xrightarrow{F^{tot}(id_{P_\bullet}, \eta)} & F^{tot}(P_\bullet, c_0(B)) = F(-, B)(P_\bullet) \\ \Big\downarrow{\scriptstyle F^{tot}(\xi, id_{Q_\bullet})} & & \Big\downarrow \\ F^{tot}(c_0(A), Q_\bullet) = F(A, -)(Q_\bullet) & \longrightarrow & F^{tot}(c_0(A), c_0(B)) = F(A, B) \end{array}$$

[35]We have two rows of long exact sequences with induced vertical maps.

the left and top maps are quasi-isomorphisms by the lemma. Taking H_i gives

$$(L_iF(A,-))(B) = H_i(F(A,Q_\bullet)) \xleftarrow{\sim} H_i(F^{tot}(P_\bullet,Q_\bullet)) \xrightarrow{\sim} H_i(F(P_\bullet,B)) = (L_iF(-,B))(A)$$

thus the theorem holds. $\square$

Remark 2.4.6. A right exact $F : \mathcal{A} \to \mathcal{B}$ is exact if and only if $L_iF = 0$ for all $i > 0$ if and only if $L_1F = 0$. [36]

Corollary 2.4.7. *Let $\mathcal{A}$ be an abelian category with enough injectives and enough projectives. Then for every $M, N \in \mathcal{A}$, we have $(R^i \operatorname{Hom}(M,-))(N) \cong (R^i \operatorname{Hom}(-,N))(M)$. In other words, if $P_\bullet \xrightarrow{\xi} M$ is a projective resolution and $N \xrightarrow{\eta} I^\bullet$ is an injective resolution, then $H^i(\operatorname{Hom}(M,I^\bullet)) \cong H^i(\operatorname{Hom}(P_\bullet,N))$.*

Proof. By Theorem 2.4.4, for right derived functors, applied to $\operatorname{Hom}_{\mathcal{A}} : \mathcal{A}^{op} \times \mathcal{A} \to Ab$. Here $\operatorname{Hom}_{\mathcal{A}}(P,-)$ (resp. $\operatorname{Hom}_{\mathcal{A}}(-,I)$) is exact for projective P (resp. injective I). [37] $\square$

<u>Notation</u> For $M, N \in \mathcal{A}$ and $i \in \mathbb{Z}$,

$$\operatorname{Ext}^i_{\mathcal{A}}(M,N) := (R^i \operatorname{Hom}(M,-))(N) \cong (R^i \operatorname{Hom}(-,N))(M)$$

<u>Long exact sequence</u> For every short exact sequence $N' \hookrightarrow N \twoheadrightarrow N''$ in $\mathcal{A}$,

$$\begin{aligned} 0 \to \operatorname{Hom}_{\mathcal{A}}(M,N') \to \operatorname{Hom}_{\mathcal{A}}(M,N) \to \operatorname{Hom}_{\mathcal{A}}(M,N'') \to \operatorname{Ext}^1_{\mathcal{A}}(M,N') \to \cdots \\ \to \operatorname{Ext}^i(M,N') \to \operatorname{Ext}^i(M,N) \to \operatorname{Ext}^i(M,N'') \to \operatorname{Ext}^{i+1}(M,N') \to \cdots \end{aligned}$$

is exact in Ab. Similarly, for every $M' \hookrightarrow M \twoheadrightarrow M''$,

$$0 \to \operatorname{Hom}(M'',N) \to \operatorname{Hom}(M,N) \to \operatorname{Hom}(M',N) \to \operatorname{Ext}^1_{\mathcal{A}}(M'',N) \to \cdots$$

is exact. (e.g. $\mathcal{A} = R\text{-}Mod$).

Corollary 2.4.8. *Let R be a ring and consider $- \otimes_R - : Mod\text{-}R \times R\text{-}Mod \to Ab$. Then for every right R-module M and left R-module N, we have*

$$(L_i(M \otimes_R -))(N) \cong (L_i(- \otimes_R N))(M)$$

Proof. This follows from the theorem because projective modules are flat : if $P \in Mod\text{-}R$ is projective, then $P \otimes_R - : R\text{-}Mod \to Ab$ is exact. This is true for $P = R$, hence true for P free ($- \otimes_R -$ commutes with $\coprod$), and also for a direct summand of a free module. [38] $\square$

<u>Notation</u> For $M \in Mod\text{-}R$, $N \in R\text{-}Mod$, $i \in \mathbb{Z}$,

$$\operatorname{Tor}^R_i(M,N) := (L_i(M \otimes_R -))(N) \cong (L_i(- \otimes_R N))(M)$$

<u>Long exact sequence</u> If $M' \hookrightarrow M \twoheadrightarrow M''$ is a short exact sequence in $Mod\text{-}R$ and $N \in R\text{-}Mod$, then we have a long exact sequence of abelian groups :

$$\begin{aligned} \cdots \to \operatorname{Tor}_{i+1}(M',N) \to \operatorname{Tor}_i(M,N) \to \operatorname{Tor}_i(M'',N) \to \operatorname{Tor}_i(M',N) \to \\ \cdots \to \operatorname{Tor}^R_1(M'',N) \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0 \end{aligned}$$

[36] If F is exact, then $H_iF(P_\bullet) = 0$ for a projective resolution $P_\bullet$. If $L_1F = 0$, then F is exact from the long exact sequence.

[37] Injectives in $\mathcal{A}^{op}$ are projective in $\mathcal{A}$!

[38] Note that $\coprod_i M_i$ is flat if and only if M_i is flat for all i. Consider $N \hookrightarrow L$ and

$$\begin{tikzcd} N \otimes (\coprod_i M_i) \arrow[r] \arrow[d, equal] & L \otimes (\coprod_i M_i) \arrow[d, equal] \\ \coprod_i (N \otimes M_i) \arrow[r] & \coprod_i (L \otimes M_i) \end{tikzcd}$$

Proposition 2.4.9. *A (right) R-module E is flat (i.e., $E \otimes_R - : R\text{-Mod} \to Ab$ is exact) if and only if* $\mathrm{Tor}_i(E, M) = 0$ *for all $M \in R$-Mod and all $i > 0$ if and only if E is $(- \otimes_R M)$-acyclic for all $M \in R$-Mod.*

Proof. E is flat (i.e., $E \otimes_R -$ is exact) if and only if $(L_i(E \otimes_R -))(M) = 0$ for all M, i (i.e., Tor $= 0$) if and only if $(L_i(- \otimes_R M))(E) = 0$ for all M, i (i.e., E is $(- \otimes_R M)$-acyclic). ☐

Example 2.4.10. If $M' \hookrightarrow M \twoheadrightarrow M''$ is exact, N is arbitrary and M'' is flat, then

$$M' \otimes_R N \hookrightarrow M \otimes_R N \twoheadrightarrow M'' \otimes_R N$$

is exact. Simply, $\mathrm{Tor}_1^R(M'', N) = 0$.

Corollary 2.4.11. *To compute* $\mathrm{Tor}_*^R(M, N)$*, it suffices to use flat resolutions. If $E_\bullet \to M$ is a resolution of M with all E_i flat, then* $\mathrm{Tor}_i^R(M, N) = H_i(E_\bullet \otimes_R N)$*. And similarly on the right.*

Proof. Theorem on the resolution by $(- \otimes_R N)$-acyclic, i.e., flat modules. ☐

Exercise 2.4.12 (Final Problem #5). Compute $\mathrm{Tor}_i^{\mathbb{Z}}(M, N)$ and $\mathrm{Ext}^i_{\mathbb{Z}}(M, N)$ for all $i \in \mathbb{Z}$ and all possible $M, N \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}\}$.

Proposition 2.4.13. *Let R be a commutative local ring ($R \setminus R^\times$ forms an ideal). Suppose R is noetherian. Let $k = R/\mathfrak{m}$. Suppose that k has a finite projective resolution (i.e., there is an exact sequence*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to k \to 0$$

with all P_i projective.) Then, every finitely generated R-module M has a finite projective resolution (i.e., R is regular).

Proof. Let M be a finitely generated R-module and let

$$0 \to N \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to M \to 0$$

be an exact sequence with all Q_i projective, finitely generated and N finitely generated (R is noetherian). It is enough to show that N is free. Observe that $\mathrm{Tor}_i(L, k) = H_i(L \otimes_R P_\bullet) = 0$ for all L and $i > n$. We claim that $\mathrm{Tor}_1(N, k) = 0$. More generally, from

$$\begin{tikzcd}
\cdots \arrow[r] & Q_{i+1} \arrow[rr] \arrow[dr, two heads] & & Q_i \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[rr] \arrow[dr, two heads] & & Q_0 \arrow[r] & M = N_0 \\
& & N_{i+1} \arrow[ur, hook] & & & & N_1 \arrow[ur, hook] & &
\end{tikzcd}$$

we have $\mathrm{Tor}_j(N_i, k) = 0$ for $j > n - i$. We use induction on i. The above observation is for $i = 0$ ($N_0 = M$). Apply Tor long exact sequence to $N_{i+1} \hookrightarrow Q_i \to N_i$:

$$0 = \mathrm{Tor}_{j+1}(Q_i, k) \to \mathrm{Tor}_{j+1}(N_i, k) \xrightarrow{\sim} \mathrm{Tor}_j(N_{i+1}, k) \to \mathrm{Tor}_j(Q_i, k) = 0$$

for $j > 0$. Hence the claim follows.
We also claim that if N is finitely generated and $\mathrm{Tor}_1(N, k) = 0$, then N is free. Pick $\overline{\alpha} : k^r \cong N/\mathfrak{m}N$ for $r \geq 1$. Take a lift $R^r \xrightarrow{\alpha} N$, then by right exactness of $- \otimes_R k$, $\operatorname{coker} \alpha \otimes_R k = \operatorname{coker} \overline{\alpha} = 0$. By Nakayama, $\operatorname{coker} \alpha = 0$. So α is an epimorphism. Consider $0 \to \ker \alpha \to R^r \xrightarrow{\alpha} N \to 0$ and

$$0 = \mathrm{Tor}_1(N, k) \to \ker \alpha \otimes_R k \to k^r \xrightarrow{\overline{\alpha}} N/\mathfrak{m}N \to 0$$

Thus $\ker \alpha \otimes_R k = 0$. By Nakayama again, $\ker \alpha = 0$, thus α is an isomorphism and $R^r \cong N$. ☐

Exercise 2.4.14 (Final Problem #6). Find a derived functor which has not been discussed in class (Tor, Ext, group (co)homology, sheaf (co)homology) and explain how it is a derived functor.

Remark 2.4.15. For modules M, N over R, there is a way to describe $\mathrm{Ext}^n_R(M, N)$ as equivalence classes of exact sequences

$$0 \to N \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Also $\mathrm{Ext}^n_R(M, N) = \mathrm{Hom}_{D(R)}(M, N[n])$ where $D(R)$ is "the derived category of R" = $K(R)[q.i.^{-1}]$.

$$\begin{tikzcd}
M: & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots \\
\bullet: & \cdots \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & N \arrow[r] \arrow[u] \arrow[d, equal] & P_{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[u] \arrow[d] & 0 \arrow[r] \arrow[u] \arrow[d] & \cdots \\
N[n]: & \cdots \arrow[r] & 0 \arrow[r] & N \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

2.5. **Group homology and cohomology.**

Let G be a group (often a finite one) and let k be a commutative ring (often $k = \mathbb{Z}$ or a field). Consider k-linear representations of G, that is, kG-modules. (Recall that kG is the "group algebra", free k-module with basis G and multiplication defined by extending k-bilinearly the rule $g \cdot h = gh$.) There is a trivial kG-module functor

$$\begin{array}{rccc} \mathrm{triv}: & k\text{-}Mod & \to & kG\text{-}Mod \\ & N & \mapsto & N = N^{\mathrm{triv}} \end{array}$$

with $g \cdot x = x$ for all $g \in G$ and $x \in N$. It has adjoints on both sides :

$$\begin{tikzcd}
kG\text{-}Mod \arrow[d, shift right=6, "(-)_G"'] \arrow[d, shift left=6, "(-)^G"] \\
k\text{-}Mod \arrow[u, "\mathrm{triv}"]
\end{tikzcd}$$

given by $M^G = \{m \in M \mid g \cdot m = m, \text{ for all } g \in G\}$ and $M_G = M/\langle gm - m \mid g \in G, m \in M\rangle$. We have $M^G \hookrightarrow M$ and $M \twoheadrightarrow M_G$. Equivalently, M^G is the biggest kG-submodule of M on which G acts trivially and M_G is the biggest quotient of M on which G acts trivially.

Remark 2.5.1. The above $\langle gm - m \mid g \in G, m \in M\rangle$ means the kG-submodule generated by $\{gm - m \mid g \in G, m \in M\}$, but it is also the abelian group generated by those $k \cdot (gm - m) = kgm - km = (kgm - m) - (km - m)$.

Definition 2.5.2. The i^{th} homology of G with coefficients in M, denoted $H_i(G, M)$ or $H^k_i(G, M)$ (very rare!), is the i^{th} derived functor of $(-)_G$ evaluated at M. The i^{th} cohomology of G with coefficients in M, denoted $H^i(G, M)$ is the i^{th} right derived functor of $(-)^G$ evaluated at M. These are k-modules.

Proposition 2.5.3. *There are natural isomorphisms :*

$$H_i(G, M) \cong \mathrm{Tor}^{kG}_i(k, M) \quad \textit{and} \quad H^i(G, M) \cong \mathrm{Ext}^i_{kG}(k, M)$$

where $k = k^{triv}$.

Proof. We have natural isomorphisms $k \otimes_{kG} M \cong M_G$[39] and $\mathrm{Hom}_{kG}(k, M) \cong M^G$[40]. Then derive! Alternatively,

$$\begin{tikzcd}[column sep=small]
 & kG\text{-}Mod \arrow[dl, phantom] & \\
\arrow[from=1-2, to=3-2, phantom] & & \\
 & k\text{-}Mod &
\end{tikzcd}$$

$$\begin{tikzcd}
kG\text{-}Mod \arrow[d, shift right=8ex, "(-)_G = k\otimes_{kG} -"'] \arrow[d, shift left=8ex, "\mathrm{Hom}_{kG}(k,-) = (-)^G"] \\
k\text{-}Mod \arrow[u, "\mathrm{Hom}_k(k,-) = \mathrm{triv} = k\otimes_k -"']
\end{tikzcd}$$

where k is considered ${}_k k_{kG}$ on the left and ${}_{kG} k_k$ on the right. ☐

Corollary 2.5.4. *For any resolution $P_\bullet \to k$ of k^{triv} by "projective" kG-modules P_i, we have*

$$H_i(G, M) = H_i(P_\bullet \otimes_{kG} M) \quad \textit{and} \quad H^i(G, M) = H^i(\mathrm{Hom}_{kG}(P_\bullet, M)) = H_{-i}(\mathrm{Hom}_{kG}(P_\bullet, M))$$

Proof. General fact about Tor and Ext. ☐

Remark 2.5.5. It is therefore enough to find one "good" projective resolution of k over kG.

Remark 2.5.6. The notation $H^i(G, M)$ does not usually involve k. The reasons are that k is usually clear from the setting, but more importantly, it does not see "restriction" (push-forward) along $k \to l$. Indeed, let $f : k \to l$ be a homomorphism of commutative rings. We have $\mathrm{res}_f : l\text{-}Mod \to k\text{-}Mod$ and $\mathrm{res}_f : kG\text{-}Mod \to lG\text{-}Mod$ which is just restriction of the scalar action from l to k via f by $x \cdot m = f(x) \cdot m$ for $x \in k$, $m \in M$ (still $g \cdot m = g \cdot m$ for $g \in G$).

Proposition 2.5.7. *With the above notation, we have natural isomorphisms*

$$H_i(G, \mathrm{res}_f M) \cong \mathrm{res}_f H_i(G, M) \quad \textit{and} \quad H^i(G, \mathrm{res}_f M) \cong \mathrm{res}_f H^i(G, M)$$

for all lG-module M.

Proof. Pick a kG-projective resolution $P_\bullet \to k$. We have

$$\begin{aligned}
\mathrm{Hom}_l({}_l l_k, M) &= \mathrm{res}_f M =_k l_l \otimes M \\
H_i(G, \mathrm{res}_f M) &= H_i(P_\bullet \otimes_{kG} (lG \otimes_{lG} M)) \\
&= H_i((P_\bullet \otimes_{kG} lG) \otimes_{lG} M) \\
&= H_i(G, M)
\end{aligned}$$

Here $P_\bullet \otimes_{kG} lG$ is an lG-projective resolution of l because $lG \otimes_{kG} - \cong l \otimes_k -$ and the sequence $P_\bullet \to k$ is split exact as k-modules[41]. Thus, $l \otimes_k P_\bullet \to l$ is a split exact sequence of l-modules, hence exact (but not split exact) as lG-modules.
For H^i, it is the same proof, using in the middle :

$$\mathrm{Hom}_{kG}(P_\bullet, \mathrm{Hom}_{lG}(lG, M)) \cong \mathrm{Hom}_{lG}(lG \otimes_{kG} P_\bullet, M).$$ ☐

Theorem 2.5.8 ((weak form of) Maschke)**.** *Let G be a finite group and k be a commutative ring. Then, the trivial kG-module k is projective as a kG-module if and only if $|G|$ is invertible in k.*

Proof. Consider $p : kG \twoheadrightarrow k$ the "augmentation" defined by $p(\sum_g a_g g) = \sum_g a_g$. So k is kG-projective if and only if p is split epimorphism of kG-modules. Consider kG-linear $\sigma : k \to kG$. It is characterized by $\sigma(1) = \sum_g a_g g$ since $x \cdot \sigma(1) = \sigma(x \cdot 1) = \sigma(x)$ for $x \in k$. We must have $a_g = a \in k$

[39] $k = kG/\langle g-1 \mid g \in G \rangle$ gives $k \otimes_{kG} M = M/\langle g-1 \mid g \in G \rangle M = M_G$
[40] $f \in \mathrm{Hom}_{kG}(k, M)$ is determined by $f(1)$ and $g \cdot f(1) = f(g \cdot 1) = f(1)$ for all $g \in G$
[41] vector spaces!

for all $g \in G$, i.e., $\sigma(1) = a \sum_g g$. The property $p \circ \sigma = id$ is equivalent to $1 = p\sigma(1) = a|G|$. This $a \in k$ exists if and only if $|G| \in k^\times$. $\square$

Corollary 2.5.9. *Let G be a finite group and M be a kG-module such that multiplication by $|G|$ is invertible on M. Then, $H_i(G, M) = 0 = H^i(G, M)$ for all $i > 0$.*

Proof. Let $l = k\left[\frac{1}{|G|}\right]$ and $f : k \to l$. Then, M is naturally an lG-module, in other words, $M = \operatorname{res}_f M =: M'$ ($S^{-1}R\text{-}Mod = R\text{-}Mod$ on which each $s \cdot -$ is invertible.) Then, $H_i(G, M) \cong H_i(G, M') = \operatorname{Tor}_i^{lG}(l, M') = 0$ (as abelian groups) for $i > 0$ since l is a projective lG-module. Similarly, $H^i(G, M) = \operatorname{Ext}^i_{lG}(l, M') = 0$ for $i > 0$. $\square$

Example 2.5.10. Let $C_2 = \langle x \mid x^2 = 1\rangle$. Then for any commutative ring k,

$$\cdots \xrightarrow{(1+x)} kC_2 \xrightarrow{(1-x)} kC_2 \xrightarrow{(1+x)} kC_2 \xrightarrow{(1-x)} kC_2 \xrightarrow{p} k \to 0$$

is a (periodic) projective resolution of k as a kC_2-module.

Exercise 2.5.11. Describe a (2-periodic) resolution of k over kC_p where $C_p = \langle x \mid x^p = 1\rangle$ for a prime p and show that $H^i(C_p, k) = k$ for all $i \geq 0$. [42]

Corollary 2.5.12 (of above Corollary)**.** *If G is finite and k is a $\mathbb{Q}$-algebra, then $H_i(G, M) = 0 = H^i(G, M)$ for all $i > 0$ and for all kG-module M.*

<u>Bar resolution</u> Let G be a group. For every $n \geq 0$, consider $P_n = kG^{(G^n)}$, the free kG-module on G^n. It has a kG-basis

$$\{[g_1 \mid g_2 \mid \cdots \mid g_n] \mid (g_1, \ldots, g_n) \in G^n\},$$

in particular, $P_0 = kG$.

A general element of P_n is a finite $\sum a_{g_1, g_2, \ldots, g_n}[g_1 \mid g_2 \mid \cdots \mid g_n]$ with $a_{g_1, g_2, \ldots, g_n} \in kG$. A k-basis of P_n is

$$\{g_0[g_1 \mid \cdots \mid g_n] \mid (g_0, g_1, \ldots, g_n) \in G^{n+1}\}.$$

For every $0 \leq i \leq n$, define $\partial_{n,i} : P_n \to P_{n-1}$ on the kG-basis by

$$\begin{aligned}
\partial_{n,0}([g_1 \mid \cdots \mid g_n]) &= g_1[g_2 \mid \cdots \mid g_n] \\
\partial_{n,i}([g_1 \mid \cdots \mid g_n]) &= [g_1 \mid \cdots \mid g_{i-1} \mid g_i g_{i+1} \mid g_{i+2} \mid \cdots \mid g_n] \text{ for } 0 < i < n \\
\partial_{n,n}([g_1 \mid \cdots \mid g_n]) &= [g_1 \mid \cdots \mid g_{n-1}]
\end{aligned}$$

Finally we let $d_n : P_n \to P_{n-1}$ to be

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_{n,i}$$

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow[{[]\mapsto 1}]{\epsilon} k \to 0$$

Lemma 2.5.13. *The above "bar resolution" $P_\bullet \xrightarrow{\epsilon} k$ is a projective resolution of k over kG.*

[42]Use $(1 + x + \cdots + x^{p-1})$ and $(1 - x)$. We also have $\operatorname{Hom}_{kC_p}(kC_p, k) = k$, $(1 - x)^* = 0$ and $(1 + x + \cdots + x^{p-1})^* = p = 0$ when $\operatorname{char} k = p$. If $\operatorname{char} k \neq p$?

Proof. Exercise to show $d^2 = 0$[43]. To show exactness of

$$\cdots \to P_n \to \cdots \to P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} k \to 0,$$

it suffices to show split exactness as a complex of k-modules. We need k-linear $e_i : P_i \to P_{i+1}$ for all $i \geq 0$ and k-linear $e_{-1} : k \to P_0$ such that $\epsilon e_{-1} = id_k$ and $d_{n+1}e_n + e_{n-1}d_n = id_{P_n}$ for all $n \geq 0$.

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r, "d_n"] & P_{n-1} \arrow[l, bend left, "e_n"] \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[l, bend left, "e_1"] \arrow[r, "\epsilon"] & k \arrow[l, bend left, "e_0"] \arrow[r] & 0 \end{tikzcd}$$

For e_{-1}, map 1 to $[]$. For $n \geq 0$, define $e_n : P_n \to P_{n+1}$ by sending the k-basis element $g_0[g_1 \mid \cdots \mid g_n]$ to $[g_0 \mid g_1 \mid \cdots \mid g_n]$. □

Exercise 2.5.14. Check $de + ed = id$.

Remark 2.5.15. Let G be a group and A be an abelian group on which G acts (i.e., A is a $\mathbb{Z}G$-module). This happens for instance if we have an extension of G by A, that is, a short exact sequence of groups

$$1 \to A \to E \xrightarrow{\pi} G \to 1$$

The G-action on A is given by ${}^g a = xax^{-1}$ for any $x \in E$ such that $\pi(x) = g$[44]. Conversely, given G and A, how many extensions E are those, as above $1 \to A \to E \to G \to 1$ up to isomorphism of extensions?

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\wr"] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

There is a well-known one : $A \rtimes G$ (= $A \times G$ with $(a, g)(b, h) = (a({}^g b), gh)$.)
Pick an extension $1 \to A \to E \xrightarrow{\pi} G \to 1$. How far is it from being split, i.e., how far is E from $A \rtimes G$? Choose a set section of π, $s : G \to E$ such that $\pi s = id$. For every $g_1, g_2 \in G$, there is a potential problem : $s(g_1 g_2) \neq s(g_1)s(g_2)$. Let $f(g_1, g_2) = s(g_1)s(g_2)s(g_1 g_2)^{-1}$. Since $\pi(f(g_1, g_2)) = 1$, we have $f(g_1, g_2) \in A$. So we have defined $f \in \mathrm{Map}(G \times G, A) \cong \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^{G^2}, A)$. Recall for the bar resolution of G over $k = \mathbb{Z}$.

$$\begin{tikzcd} \cdots \arrow[r] & P_n \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P_3 \arrow[r, "d_3"] \arrow[d, equal] & P_2 \arrow[r, "d_2"] \arrow[d, equal] & P_1 \arrow[r, "d_1"] \arrow[d, equal] & P_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\ & (\mathbb{Z}G)^{G^n} & & (\mathbb{Z}G)^{G^3} & (\mathbb{Z}G)^{G^2} & (\mathbb{Z}G)^{G} & & & \end{tikzcd}$$

$$[g_1 \mid g_2] \xmapsto{d_2} g_1[g_2] - [g_1 g_2] + [g_1]$$

$$[g_1 \mid g_2 \mid g_3] \xmapsto{d_3} g_1[g_2 \mid g_3] - [g_1 g_2 \mid g_3] + [g_1 \mid g_2 g_3] - [g_1 \mid g_2]$$

[43] We have $\partial_{n-1,0}\partial_{n,0} = \partial_{n-1,0}\partial_{n,1}$ and $\partial_{n-1,0}\partial_{n,i} = \partial_{n-1,i-1}\partial_{n,0}$ for $1 < i < n$, etc.
[44] This makes sense because $xax^{-1} \in \ker \pi = A$

We have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \arrow[r, "d_1^*"] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^G, A) \arrow[r, "d_2^*"] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^{G^2}, A) \arrow[r, "d_3^*"] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}G}((\mathbb{Z}G)^{G^3}, A) \arrow[d, equal] \\
& A & \mathrm{Map}(G, A) & \mathrm{Map}(G^2, A) \arrow[r, "\text{explicit}"] & \mathrm{Map}(G^3, A) \\
& & & \Cup & \Cup \\
& & & f \arrow[r, mapsto] & d_3^* f
\end{tikzcd}$$

where $(d_3^* f)(g_1, g_2, g_3) = {}^{g_1} f(g_2, g_3)\{f(g_1 g_2, g_3)\}^{-1} f(g_1, g_2 g_3)\{f(g_1, g_2)\}^{-1}$.
Back to our extension $1 \to A \to E \underset{s}{\overset{\pi}{\rightleftharpoons}} G \to 1$. Our function $f = f_s$ with

$$f_s(g_1, g_2) = s(g_1)s(g_2)s(g_1 g_2)^{-1} \in A$$

belongs to the kernel of $d_3^* : \mathrm{Map}(G^2, A) \to \mathrm{Map}(G^3, A)$[45]. Thus it defines a class $[f_s] \in H^2(\mathrm{Map}(G^\bullet, A)) = H^2(G, A)$. The dependency of $[f_s]$ on s disappears in H^2! Another choice of s' yields some $h \in \mathrm{Map}(G, A)$ such that $d_2^* h = f_s - f_{s'}$.

Theorem 2.5.16. *We keep notations as above. In particular, A is a given $\mathbb{Z}G$-module (the G-action on A is fixed.) The above construction yields a bijection between the isomorphism classes of extensions $1 \to A \to E \to G \to 1$ and $H^2(G, A)$. In particular, $[f_s] = 0$ if and only if $E \cong A \rtimes G$ (as an extension).*

Proof. Long verification. Given $[f] \in H^2(G, A)$, one can construct an extension $E_f = A \times G$ with

$$(a, g) *_f (b, h) = (a + {}^g b + f(g, h), gh).$$ □

2.6. Sheaf cohomology.

<u>**Setup**</u> Let X be a topological space and $Sh(X)$ be the category of sheaves of abelian groups (or generalizations). We know that $Sh(X)$ has enough injectives. ($F \hookrightarrow \prod_{x \in X} (i_x)_* I(F_x)$ where $I(A) = \prod_{\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z}$.) Recall that $\Gamma(X, -) : Sh(X) \to Ab$ is only left exact.

Definition 2.6.1. Let $F \in Sh(X)$. The i^{th} right derived functor of $\Gamma(X, -)$ evaluated at F is the i^{th} cohomology group of X with coefficients in F.

$$H^i(X, F) := (R^i \Gamma(X, -))(F)$$

Take an injective resolution $F \to I^\bullet$ of F in $Sh(X)$. Then,

$$H^i(X, F) = H^i(\Gamma(X, I^\bullet))$$

for all $i \in \mathbb{Z}$. In particular, $H^0(X, F) = \Gamma(X, F) = F(X)$.
From the general theory, for every short exact sequence of sheaves,

$$0 \to F' \to F \to F'' \to 0,$$

we have a long exact sequence of abelian groups

$$0 \to F'(X) \to F(X) \to F''(X) \xrightarrow{\partial} H^1(X, F') \to \cdots$$

[45] A is abelian!

Definition 2.6.2. A sheaf $E \in Sh(X)$ is called flasque (flabby) if for every open $V \subseteq U \subseteq X$, the restriction $E(U) \to E(V)$ is onto.

Proposition 2.6.3. *(1) Injectives are flasque.*

(2) If $0 \to E \to F \to F' \to 0$ is exact in $Sh(X)$ and E is flasque, then $0 \to E(X) \to F(X) \to F'(X) \to 0$ is exact.

(3) Flasque sheaves are $\Gamma(X,-)$-acyclic : if E is flasque, then $H^i(X,E) = 0$ for all $i > 0$.

(4) Every sheaf F admits a monomorphism $F \hookrightarrow \prod_{x\in X}(i_x)_*(F_x) =: E_F$ with E_F flasque. In cash, $E_F(U) = \prod_{x\in U} F_x$.

Proof. (1) For every open $U \subseteq X$, consider $\underline{\mathbb{Z}}_U$ = the sheafification of the presheaf

$$W \mapsto \begin{cases} \mathbb{Z} & \text{if } W \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

($\underline{\mathbb{Z}}_U = j_!\mathcal{O}_U$). Two facts : if $V \subseteq U$, then $\underline{\mathbb{Z}}_V \hookrightarrow \underline{\mathbb{Z}}_U$.

$$\mathrm{Hom}_{Sh(X)}(\underline{\mathbb{Z}}_U, F) = \mathrm{Hom}_{PreSh(X)}(\mathbb{Z}_U^{pre}, F) \cong \mathrm{Hom}_{Ab}(\mathbb{Z}, F(U)) \cong F(U)$$

Also,

$$\begin{tikzcd} \mathrm{Hom}_{Sh(X)}(\underline{\mathbb{Z}}_U, F) \arrow[r, "\sim"] \arrow[d] & F(U) \arrow[d, "\mathrm{res}_{U,V}"] \\ \mathrm{Hom}_{Sh(X)}(\underline{\mathbb{Z}}_V, F) \arrow[r, "\sim"] & F(V) \end{tikzcd}$$

if $V \subseteq U$. If F is injective, then the left vertical map is surjective[46]. Hence, F is flasque.

(2) Let $0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} F' \to 0$ be exact and E be flasque. We want to show that $\beta : F(X) \to F'(X)$ is onto. Pick $t \in F'(X)$ and let's construct $s \in F(X)$ such that $\beta(s) = t$. The assumption implies that t is in the image of β locally, around every point.

On $\{(U,s) \mid U \subseteq X \text{ open}, s \in F(U), \beta(s) = t|_U\}$, we set $(U,s) \leq (U',s')$ if $U \subseteq U'$ and $s'|_U = s$. Since F is a sheaf, there exists by Zorn's lemma a maximal such (U,s). We claim that $U = X$. Otherwise, pick $x \in X \setminus U$, $x \in V \subseteq X$ open, and $s' \in F(V)$ such that $s' \overset{\beta}{\mapsto} t|_V$. To define $\widehat{s} \in F(U \cup V)$ by gluing $s \in F(U)$ and $s' \in F(V)$, we would need $s|_{U\cap V} = s'|_{U\cap V}$. In fact, $s|_{U\cap V} - s'|_{U\cap V} \overset{\beta}{\mapsto} t|_{U\cap V} - t|_{U\cap V} = 0$. Hence there exists $r \in E(U \cap V)$ such that $\alpha(r) = s|_{U\cap V} - s'|_{U\cap V}$. Since E is flasque, there exists $F' \in E(V)$ such that $r'|_{U\cap V} = r$. Then correct $s' \in F(V)$ by r', that is $s'' = s' + \alpha(r') \in F(V) \overset{\beta}{\mapsto} t|_V$. Now, by construction, $s|_{U\cap V} = s''|_{U\cap V}$. Hence there exists $\widehat{s} \in F(U \cup V)$ such that $\widehat{s}|_U = s \mapsto t|_U$ and $\widehat{s}|_V = s'' \mapsto t|_V$. Hence $\widehat{s} \mapsto t|_{U\cup V}$ (because F is a sheaf.) Hence $(U,s) \lneq (U \cup V, \widehat{s})$, which is a contradiction. So $U = X$.

(3) Let E be flasque and let $0 \to E \to I \to F \to 0$ be exact with I injective. Then,

$$0 \to E(X) \to I(X) \to F(X) \to H^1(X,E) \to H^1(X,I) = 0 \to \cdots$$

So $H^1(X,E) = 0$ (by (2)) and $H^{i+1}(X,E) = H^i(X,F)$ for all $i \geq 1$. It suffices to show that F is flasque. More generally, if $0 \to E \to E' \to F \to 0$ is exact and E, E' are flasque, then F is flasque. Since E is flasque, $E|_U$ is also flasque. So, $0 \to E|_U \to E'|_U \to F|_U \to 0$ is exact. By (2),

[46] $\mathrm{Hom}_{Sh(X)}(-,F)$ is exact!

$0 \to E(U) \to E'(U) \to F(U) \to 0$ is exact. For $V \subseteq U$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E(U) \arrow[r] \arrow[d, two heads, "\mathrm{res}_{U,V}"] & E'(U) \arrow[r] \arrow[d, two heads, "\mathrm{res}_{U,V}"] & F(U) \arrow[r] \arrow[d, "\mathrm{res}_{U,V}"] & 0 \\ 0 \arrow[r] & E(V) \arrow[r] & E'(V) \arrow[r] & F(V) \arrow[r] & 0 \end{tikzcd}$$

This shows that the right vertical map $\mathrm{res}_{U,V} : F(U) \to F(V)$ is onto.
(4) $F \to E_F$ is injective "stalk-wise" and E_F is clearly flasque.

$$\begin{tikzcd} E_F(U) \arrow[r, equal] \arrow[d] & \prod_{x \in U} F_x \arrow[d, two heads] \\ E_F(V) \arrow[r, equal] & \prod_{x \in V} F_x \end{tikzcd}$$

□

Corollary 2.6.4. *If* $0 \to F \to E^0 \to E^1 \to \cdots \to E^n \to E^{n+1} \to \cdots$ *is exact with all* E^i *flasque, then* $H^i(X, F) = H^i(E^\bullet(X))$.

3. Spectral sequences (an introduction)

Reference for more : J. McClear y "A User's Guide to Spectral Sequences"
For the whole chapter, there is fixed abelian category $\mathcal{A}$ (satisfying some axioms, for convergence issues). e.g. $\mathcal{A} = R\text{-}Mod$ for some ring R.

3.1. **Introduction.**

Recall that if $A'_\bullet \hookrightarrow A_\bullet \twoheadrightarrow A_\bullet/A'_\bullet$ is an exact sequence in $Ch(\mathcal{A})$, then we have a long exact sequence in homology :

$$\cdots \to H_i(A'_\bullet) \to H_i(A_\bullet) \to H_i(A_\bullet/A'_\bullet) \to H_{i-1}(A'_\bullet) \to \cdots$$

We thus have some control ("homological") of A, or rather $H_*(A)$, once we know $H_*(A')$ and $H_*(A/A')$ - think of the latter as "known" and $H_*(A)$ as unknown. More precisely, there exist maps

$$H_*(A/A') \xrightarrow{\partial} H_{*-1}(A')$$

which yield some (known) objects $\ker \partial$ and $\operatorname{coker} \partial$. Then $H_*(A)$ has a (one-step) filtration $H_i(A) \supseteq J_i \supseteq 0$ such that $H_i(A)/J_i \cong \ker \partial$ and $J_i/0 \cong \operatorname{coker} \partial$ where
$$\begin{tikzcd} H_i(A'_\bullet) \arrow[rr] \arrow[dr, two heads] & & H_i(A_\bullet) \\ & J_i \arrow[ur, hook] & \end{tikzcd}.$$

Exercise 3.1.1. Suppose $0 \subseteq A'' \subseteq A' \subseteq A$ subcomplexes. Think $A''/0$, A'/A'' and A/A' are known. How to get $H_*(A)$ from $H_*(A''/0)$, $H_*(A'/A'')$ and $H_*(A/A')$?

Definition 3.1.2. A (homological) spectral sequence starting on s^{th} page (s is usually 0,1, or 2) is a collection $(E^r_{p,q}, d^r_{p,q})_{r \geq s, (p,q) \in \mathbb{Z}}$ where $E^r_{p,q}$ is an object in $\mathcal{A}$ and $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ (total degree goes down by 1) such that $d^r d^r = 0$ together with isomorphisms

$$E^{r+1}_{p,q} \cong H(E^r_{p+r,q-r+1} \xrightarrow{d^r} E^r_{p,q} \xrightarrow{d^r} E^r_{p-r,q+r-1}) = \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}}.$$

(Pictures) $\underline{s = 1}$

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ \cdots & \longleftarrow & E^1_{0,1} & \xleftarrow{d^1_{1,1}} & E^1_{1,1} & \longleftarrow & \cdots \\ \cdots & \longleftarrow & E^1_{0,0} & \xleftarrow{d^1_{1,0}} & E^1_{1,0} & \longleftarrow & \cdots \\ & & \vdots & & \vdots & & \end{array}$$

with $E^1_{p-1,q} \xleftarrow{d^1_{p,q}} E^1_{p,q}$. Every line is a complex, $d^1 d^1 = 0$.

$\underline{s=2}$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \bullet & \bullet & \bullet & \bullet & \cdots \\
\cdots & \bullet & \bullet & \bullet \arrow[ull] & \bullet \arrow[ull] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with $E^2_{p-2,q+1} \xleftarrow{d^2_{p,q}} E^2_{p,q}$.

Remark 3.1.3. Cohomology spectral sequences are same : $(E_r^{p,q}, d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1})$ with $E_{r+1} \cong H(E_r, d_r)$.

Remark 3.1.4. $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$, hence they are all subquotients of $E^s_{p,q}$. Hence $E^r_{p,q} \cong Z^r_{p,q}/B^r_{p,q}$ where

$$0 = B^s_{p,q} \subseteq B^{s+1}_{p,q} \subseteq \cdots \subseteq B^r_{p,q} \subseteq \cdots \subseteq Z^r_{p,q} \subseteq \cdots \subseteq Z^{s+1}_{p,q} \subseteq Z^s_{p,q} = E^s_{p,q}$$

Definition 3.1.5. With the above notation, $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$ where

$$Z^\infty_{p,q} = \bigcap_{r \geq s} Z^r_{p,q}(\text{limit}), \quad B^r_{p,q} = \bigcup_{r \geq s} B^r_{p,q}(\text{colimit})$$

Remark 3.1.6. We say that a spectral sequence collapses at place (p,q) at page r_0 if $d^r_{p,q} = 0$ and $d^r_{p+r,q-r+1} = 0$ for all $r \geq r_0$. In that case, $E^{r_0}_{p,q} \cong E^{r_0+1}_{p,q} \cong \cdots \cong E^r_{p,q} \cong E^\infty_{p,q}$ for all $r \geq r_0$.

Example 3.1.7. If the spectral sequence is a first quadrant spectral sequence, i.e., $E^r_{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$, then it collapses at every place at some corresponding page.

Definition 3.1.8. A spectral sequence $(E^r_{p,q})_{r \geq s}$ weakly converges towards a collection of objects $(H_n)_{n \in \mathbb{Z}}$ if there exist filtrations

$$\cdots \subseteq J_{p-1,n} \subseteq J_{p,n} \subseteq J_{p+1,n} \subseteq \cdots \subseteq H_n$$

such that $J_{p,n}/J_{p-1,n} \cong E^\infty_{p,n-p}$. (Note that $q = n-p$, that is, $p+q = n$.)

<u>Notation</u> : $E^s_{p,q} \underset{n=p+q}{\Longrightarrow} H_n$

e.g. $E^2_{p,q} = (\text{known stuff}) \ \Rightarrow \ H_{p+q} = (\text{mysterious stuff})$

Remark 3.1.9. The above doesn't say that H_n is exhausted by the filtration. ($\bigcup_p J_{p,n} = H_n$? and $\bigcap_p J_{p,n} = 0$?) Meditate $\cdots \subseteq 2^n\mathbb{Z} \subseteq \cdots \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$. Even if it exhausts, the information about H_* can be weak. (all $J_p/J_{p-1} = \mathbb{Z}/2\mathbb{Z}$, but $H = \mathbb{Z}$ is quite different.)

Definition 3.1.10. A spectral sequence $(E^r_{p,q}, d^r_{p,q})$ is bounded below if for every (total degree) n, there exists $p_0 = p_0(n)$ such that $E^s_{p,n-p} = 0$ for all $p \leq p_0(n)$ (thus $E^r_{p,n-p} = 0$ for all $r \geq s$.)

Definition 3.1.11 (Bounded-below convergence)**.** A bounded below spectral sequence converges to $(H_n)_{n\in\mathbb{Z}}$ if it weakly converges, i.e.,

$$\cdots \subseteq J_{p-1,n} \subseteq J_{p,n} \subseteq \cdots \subseteq H_n$$

such that $J_{p,n}/J_{p-1,n} \cong E^{\infty}_{p,n-p}$ and moreover, $\bigcap\limits_p J_{p,n} = 0$ (if and only if $J_{p,n} = 0$ for $p \ll 0$) and $\bigcup\limits_p J_{p,n} = H_n$.

3.2. **Exact couples.**

Definition 3.2.1. An exact couple $(D, E, \alpha, \beta, \gamma)$ is an exact sequence $\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$ (i.e., exact at D, at D, and at E). Note that $d = \beta\gamma : E \to E$ satisfies $dd = 0$.

Proposition 3.2.2. *Let* $\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$ *be an exact couple. Let $D' = \operatorname{im}\alpha$ and $E' = H(E, d) = \ker\beta\gamma/\operatorname{im}\beta\gamma$. Let $\alpha' : D' \to D'$ be the restriction of α and $\gamma' : E' \to D'$ be the morphism induced by γ. (on elements, $\gamma'([x]) = \gamma(x)$)*

$$\begin{tikzcd} D' = \operatorname{im}\alpha \arrow[rr, "\alpha'"] & & D' \arrow[dl, "\beta'"] \\ & E' \arrow[ul, "\gamma'"] & \end{tikzcd}$$

Let $\beta' : D' \to E'$ be "$\beta' = [\beta \circ \alpha^{-1}]$" which means on elements $\beta'(y) = [\beta(x)] \in E'$ for any $x \in B$ such that $y = \alpha(x)$. Since $y \in D' = \operatorname{im}\alpha$, we have $y = \alpha(x)$ for same $x \in D$.

Then, these morphisms are well-defined and $\begin{tikzcd} D' \arrow[rr, "\alpha'"] & & D' \arrow[dl, "\beta'"] \\ & E' \arrow[ul, "\gamma'"] & \end{tikzcd}$ *is again exact.*

Proof. Well-definedness is easy. Exactness is an exercise. For instance, if $x \in E'$ such that $\gamma'(x) = 0$, then $x = [t] \in \ker\beta\gamma/\operatorname{im}\beta\gamma$ where $t \in E$ and $\beta\gamma(t) = 0$. We have $\gamma(t) = 0$, i.e., $t \in \ker\gamma = \operatorname{im}\beta$, so $t = \beta(u)$ for $u \in D$. Let $y = \alpha(u) \in \operatorname{im}\alpha = D'$, then $\beta'(y) = [\beta(u)] = [t] = x$. $\square$

Remark 3.2.3. Given an exact couple $\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$, $\begin{tikzcd} D' \arrow[rr, "\alpha'"] & & D' \arrow[dl, "\beta'"] \\ & E' \arrow[ul, "\gamma'"] & \end{tikzcd}$ is called the derived exact couple. By induction, we get a tower of exact couples

$$(D, E, \alpha, \beta, \gamma) \xrightarrow{(-)'} (D', E', \alpha', \beta', \gamma') \xrightarrow{(-)'} \cdots \xrightarrow{(-)'} (D^{(t)}, E^{(t)}, \alpha^{(t)}, \beta^{(t)}, \gamma^{(t)})$$

Lemma 3.2.4. *For every $t \geq 1$,*

$$D^{(t)} = \operatorname{im}\alpha^{(t)}, \quad \alpha^{(t)} = \alpha, \quad E^{(t)} = Z^{(t)}/B^{(t)}$$

where $B^{(t)} \subseteq Z^{(t)} \subseteq E$ are given by

$$Z^{(t)} = \gamma^{-1}(\operatorname{im}\alpha^t), \quad B^{(t)} = \beta(\ker \alpha^t)$$

and $\gamma^{(t)} = \gamma|_{\dots}$ and $\beta^{(t)} = [\beta \circ \alpha^{-t}]$.

Proof. Exercise. $\square$

Lemma 3.2.5. *Let $D_{\bullet\bullet}$ and $E_{\bullet\bullet}$ be $\mathbb{Z}^2$-bigraded objects (collection of $D_{p,q}$ for $(p,q) \in \mathbb{Z}^2$). Let*
$$\begin{tikzcd} D_{\bullet\bullet} \arrow[rr, "\alpha"] & & D_{\bullet\bullet} \arrow[dl, "\beta"] \\ & E_{\bullet\bullet} \arrow[ul, "\gamma"] & \end{tikzcd}$$
be an exact couple of $\mathbb{Z}^2$-graded objects with α of bidegree $(1,-1)$, β of bidegree $(-b,b)$ and γ of bidegree $(-1,0)$. Then, the derived couple
$$\begin{tikzcd} D'_{\bullet\bullet} \arrow[rr, "\alpha'"] & & D'_{\bullet\bullet} \arrow[dl, "\beta'"] \\ & E'_{\bullet\bullet} \arrow[ul, "\gamma'"] & \end{tikzcd}$$
has bidegrees $(1,-1)$ for α', $(-b-1, b+1)$ for β' and $(-1,0)$ for γ'.

Proof. Easy. $\operatorname{bideg}(\beta') = \operatorname{bideg}(\beta) - \operatorname{bideg}(\alpha)$, etc. $\square$

Corollary 3.2.6. *Let $(D^r, E^r, \alpha^r, \beta^r, \gamma^r)$ be a collection of exact couples for $r \geq s$ such that*

$$(D^{r+1}, E^{r+1}, \cdots) = (D^r, E^r, \cdots)'$$

(i.e., we give $(D, E, \cdots) = (D^s, E^s, \cdots)$ and $(D^r, E^r, \cdots) = (D, E, \cdots)^{(r-s)}$.) Suppose that $\alpha = \alpha^s$ has bidegree $(1,-1)$, $\gamma = \gamma^s$ has bidegree $(-1,0)$ and $\beta = \beta^s$ has bidegree $(-s+1, s-1)$ (typically $(0,0)$ if we start on $s = 1$). Then, $(E^r_{\bullet\bullet}, d^r = \beta^r\gamma^r)$ is a spectral sequence starting on page s.

Definition 3.2.7. Like for spectral sequences, an exact couple $(D_{\bullet\bullet}, E_{\bullet\bullet}, \dots)'$ is bounded below if for every $n \in \mathbb{Z}$, there is $p_0 = p_0(n)$ such that $D_{p,n-p} = 0$ for $p \leq p_0$ (thus, $E_{p,n-p} = 0$ for $p \ll 0$.) In that case, the associated spectral sequence is bounded below.

Theorem 3.2.8. *Let*
$$\begin{tikzcd} D_{\bullet\bullet} \arrow[rr, "\alpha"] & & D_{\bullet\bullet} \arrow[dl, "\beta"] \\ & E_{\bullet\bullet} \arrow[ul, "\gamma"] & \end{tikzcd}$$
be an exact couple with bidegrees $(1,-1), (-s+1, s-1), (-1,0)$ for α, β, γ and let $(E^r_{p,q}, d^r)_{r \geq s}$ be the associated spectral sequence. Suppose that the exact couple is bounded below. Let

$$H_n = \operatorname{colim}_{p \to +\infty}(D_{p,n-p}, \alpha) = \operatorname{colim}(D_{p,n-p} \xrightarrow{\alpha} D_{p+1,n-p-1} \xrightarrow{\alpha} \cdots)$$

Then, the bounded below sequence $E^s_{p,q} \underset{n=p+q}{\Longrightarrow} H_n$ converges to that H_*.

Proof. The filtration on H_n is given by

$$\cdots \subseteq J_{p-1,n} \subseteq J_{p,n} \subseteq \cdots \subseteq H_n$$

where $J_{p,n} = \operatorname{im}(D_{p+s-1,n-p-s+1} \to \operatorname{colim}_{i \to \infty} D_{i,n-i} = H_n)$. This filtration exhausts H_n because the couple is bounded below. We need to give isomorphisms

$$J_{p,n}/J_{p-1,n} \cong E^\infty_{p,n-p} = Z^\infty_{p,q}/B^\infty_{p,q} \quad (q = n-p)$$

where $Z^\infty_{p,q} = \bigcap_r Z^r_{p,q}$ and $B^r_{p,q} = \bigcup_r B^r_{p,q} \subseteq E_{p,q}$.

Recall that $E^r_{p,q} = \dfrac{\gamma^{-1}(\operatorname{im}\alpha^{r-s})}{\beta(\ker \alpha^{r-s})}$ or more precisely, $Z^r_{p,q} = \gamma^{-1}(\operatorname{im}\alpha^{r-s})$ and $B^r_{p,q} = B(\ker \alpha^{r-s})$.

$$Z^r_{p,q} = \gamma^{-1}(\operatorname{im}(\alpha^{r-s} : D_{p-r+s-1,q+r-s} \to D_{p-1,q}))$$

$$\xRightarrow[D_{i,n-1-i}=0 \text{ for } i\ll 0]{} Z^\infty_{p,q} = \bigcap Z^r_{p,q} = \ker(\gamma : E_{p,q} \to D_{p-1,q})$$

For each p, q such that $p+q=n$, consider

$$0 \to K_{p+s-1,n-p-s+1} \to D_{p+s-1,n-p-s+1} \to J_{p,n} \to 0$$

Compare two consecutive sequences.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & K_{p-1,\dots} \arrow[r] \arrow[d] & D_{p-1+s-1,\dots} \arrow[r, "{''\alpha^\infty{''}}"] \arrow[d] & J_{p-1,n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{p,\dots} \arrow[r] \arrow[d, two heads] & D_{p+s-1,\dots} \arrow[r, "{''\alpha^{\infty-1}{''}}"] \arrow[d, two heads, "\beta", "(2)"'] & J_{p,n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, "(1)"] & \beta(K_{p+s-1,\dots})(3) \arrow[r] & \beta(D_{p+s-1,\dots}) \arrow[r] \arrow[d, equal] & J_{p,n}/J_{p-1,n} \arrow[r] \arrow[d] & 0 \\
 & & \ker(\gamma : E_{p,q} \to D_{p-1,q}) = Z^\infty_{p,q} & 0 &
\end{tikzcd}$$

(1) Apply Snake.
(2) By the exact couple, coker $\alpha = \operatorname{im}\beta = \beta(D_{\dots})$.
(3) By (1) and (2).
By construction,

$$\beta(K_{p+s-1,\dots}) = \bigcup_{t\geq 1} \beta(\ker \alpha^t) = \bigcup_{r\geq 1} B^r_{p,q} = B^\infty_{p,q} \qquad \square$$

3.3. **Some examples.**
Spectral sequence of a filtered complex Let

$$\cdots \subseteq F_{p-1}C_\bullet \subseteq F_pC_\bullet \subseteq \cdots \subseteq C_\bullet$$

be a filtration by subcomplexes. Suppose the filtration is bounded below : for all $n \in \mathbb{Z}$, $F_pC_n = 0$ for $p \ll 0$. Suppose $C_n = \bigcup_{p\in\mathbb{Z}} F_{p,n}$. Then, there exists a bounded below converging spectral sequence

$$E^1_{p,q} = H_{p,q}(F_pC_\bullet/F_{p-1}C_\bullet) \xRightarrow[p+q=n]{} H_n(C_\bullet)$$

Proof. Let $D_{p,q} = H_{p+q}(F_pC_\bullet)$ and $E_{p,q} = H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet)$. There is a long exact sequence in H_* on $F_{p-1} \hookrightarrow F_p \twoheadrightarrow F_p/F_{p-1}$.

$$\begin{tikzcd}
D_{\bullet\bullet} \arrow[rr, "\alpha", "{(1,-1)}"'] & & D_{\bullet\bullet} \arrow[dl, "\beta", "{(0,0)}"'] \\
 & E_{\bullet\bullet} \arrow[ul, "\gamma", "{(-1,0)}"'] &
\end{tikzcd} \qquad \square$$

Spectral sequence of a double complex   If $C_{\bullet\bullet}$ is 1^{st} quadrant ($C_{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$) double complex, then

$$^{I}E^2_{p,q} = H^h_p(H^v_q(C_{\bullet\bullet})) \Longrightarrow H_{p+q}(Tot^{\oplus}(C_{\bullet\bullet}))$$

and same for $^{II}E^2_{p,q} = H^v_p H^h_q(C_{\bullet\bullet})$.

Grothendieck spectral sequence   Suppose $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ and F, G are both right exact. If $F(\text{proj}) \subseteq G$-acyclic, then

$$E^2_{p,q} = (L_p G)(L_q F)(A) \Longrightarrow L_{p+q}(GF)(A).$$