Algebraic K-Theory in Low Degrees

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1 The Grothendieck group of an abelian monoid

Prerequisites

Abelian monoids

Recall that an abelian monoid is a set M together with a binary operation $\oplus : M \times M \longrightarrow M$ and a distinguished element 0 such that the following axioms hold true.

(AMon1) The operation $\oplus$ is associative that means $m_1 \oplus (m_2 \oplus m_3) = (m_1 \oplus m_2) \oplus m_3$ for all $m_1, m_2, m_3 \in M$,

(AMon2) The element 0 is neutral with respect to $\oplus$ that means $0 \oplus m = m \oplus 0 = m$ for all $m \in M$,

(AMon3) The operation $\oplus$ is commutative that means $m_1 \oplus m_2 = m_2 \oplus m_1$ for all $m_1, m_2 \in M$.

The category AMon of abelian monoids is a full subcategory of the category of monoids. Morphisms of AMon are given by maps $f : M \to \tilde{M}$ between abelian monoids M and $\tilde{M}$ with binary operations $\oplus$ and $\tilde{\oplus}$, respectively, such that the following axiom holds true.

(MorMon) For all $m_1, m_2 \in M$ the relation $f(m_1 \oplus m_2) = f(m_1) \tilde{\oplus} f(m_2)$ holds true.

Objective

The category AGrp of abelian groups is a full subcategory of AMon. The main goal of the following considerations is to construct a left adjoint to the embedding functor $\iota : \mathsf{AGrp} \hookrightarrow \mathsf{AMon}$.

Construction of the Grothendieck group

Definition 1.1. Let M be an abelian monoid. An abelian group K together with a morphism $\kappa : M \to K$ of monoids is called a *Grothendieck group* of M, if the following universal property is satisfied:

(Gro) For every abelian group A and every morphism of monoids $f : M \to A$ there exists a unique homomorphism of groups $f_K : K \to A$ such that the following diagram commutes.

$$
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\kappa"'] & A \\
K \arrow[ur, "f_K"'] &
\end{tikzcd}
\tag{1.1}
$$

Clearly, if a Grothendieck group exists for M, then it is unique up to isomorphism by the universal property. Let us show that for every ablian monoid M there exists a Grothendieck group. To this end let $\mathsf{F}(M)$ be the free abelian group generated by the elements of M, and denote for every $m \in M$ by $\overline{m}$ the image of m in $\mathsf{F}(M)$ under the canonical injection $M \to \mathsf{F}(M)$. Let $\mathsf{R}(M) \subset \mathsf{F}(M)$ be the (necessarily free) subgroup generated by all elements of the form $\overline{m_1 \oplus m_2} - \overline{m}_1 - \overline{m}_2$, where $m_1, m_2 \in M$ and where $-$ denotes subtraction within the abelian group $\mathsf{F}(M)$. Then the following holds true.

Proposition 1.2. *For every abelian monoid M, the abelian group $\mathsf{K}^{\mathrm{Gro}}(M) := \mathsf{F}(M)/\mathsf{R}(M)$ together with the canonical morphism of monoids $\kappa_M^{\mathrm{Gro}} : M \to \mathsf{K}^{\mathrm{Gro}}(A)$, $m \mapsto \overline{m} + \mathsf{R}(M)$ is a Grothendieck group for M.*

Proof. Let A be an abelian group and $M \to A$ a morphism of monoids. By the universal property of $\mathsf{F}(M)$, there exists a unique group homomorphism $f_{\mathsf{F}(M)} : F(M) \to A$ such that the diagram

$$
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d] & A \\
\mathsf{F}(M) \arrow[ur, "f_{\mathsf{F}(M)}"'] &
\end{tikzcd}
\tag{1.2}
$$

commutes, where $M \to \mathsf{F}(M)$ is the canonical embedding. Observe now that for all $m_1, m_2 \in M$

$$
\begin{aligned}
f_{\mathsf{F}(M)}(\overline{m_1 \oplus m_2} - \overline{m_1} - \overline{m_2}) &= f_{\mathsf{F}(M)}(\overline{m_1 \oplus m_2}) - f_{\mathsf{F}(M)}(\overline{m_1}) - f_{\mathsf{F}(M)}(\overline{m_2}) \\
&= f(m_1 \oplus m_2) - f(m_1) - f(m_2) \\
&= f(m_1) + f(m_2) - f(m_1) - f(m_2) = 0,
\end{aligned}
$$

hence $f_{\mathsf{F}(M)}$ factorizes through the map $\mathsf{F}(M) \to \mathsf{K}^{\mathrm{Gro}}(M)$. In other words this means that there exists a homomorphism $f_{\mathsf{K}^{\mathrm{Gro}}(M)} : \mathsf{K}^{\mathrm{Gro}}(M) \to A$ such that the diagram

$$
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d] & A \\
\mathsf{F}(M) \arrow[ur, "f_{\mathsf{F}(M)}"] \arrow[d] & \\
\mathsf{K}^{\mathrm{Gro}}(M) \arrow[uur, "f_{\mathsf{K}^{\mathrm{Gro}}(M)}"'] &
\end{tikzcd}
\tag{1.3}
$$

commutes. By the universal property of the free abelian group $\mathsf{F}(M)$, the homomorphism $f_{\mathsf{F}(M)}$ is uniquely determined by f. Since $\mathsf{F}(M) \to \mathsf{K}^{\mathrm{Gro}}(M)$ is an epimorphism, $f_{\mathsf{K}^{\mathrm{Gro}}(M)}$ is uniquely determined by $f_{\mathsf{F}(M)}$, hence uniqueness of $f_{\mathsf{K}^{\mathrm{Gro}}(M)}$ follows. This proves the claim, since the composition of the two vertical arrows in Diagram (1.3) coincides with κ_M^{Gro}. □

From now on, we will denote by $[m]$ the equivalence class of an element $m \in M$ in the Grothendieck group $\mathsf{K}^{\mathrm{Gro}}(M)$. As we will see later, the map $M \to \mathsf{K}^{\mathrm{Gro}}$, $m \mapsto [m]$ need not be injective, in general.

Another representation of the Grothendieck group

Next, let us provide a second representation for $\mathsf{K}^{\mathrm{Gro}}$. To this end consider the map

$$\lambda : \ M \times M \to \mathsf{K}^{\mathrm{Gro}}, \ \ (m_1, m_2) \mapsto [m_1] - [m_2].$$

By construction of $\mathsf{K}^{\mathrm{Gro}}$, this map must be surjective. Note that $M \times M$ inherits the structure of a commutative monoid from M. Let us determine, when $\lambda(m_1, n_1) = \lambda(m_2, n_2)$ for $m_1, m_2, n_1, n_2 \in M$. The following observation is crucial for this.

Lemma 1.3. *For all $m_1, m_2 \in M$ one has $[m_1] = [m_2]$ in $\mathsf{K}^{\mathrm{Gro}}$ if and only if there is an $n \in M$ such that $m_1 \oplus n = m_2 \oplus n$.*

Definition 1.4. Two elements m_1, m_2 of an abelian monoid M are called *stably equivalent*, if there is an $n \in M$ such that $m_1 \oplus n = m_2 \oplus n$.

Proof of the Lemma. If m_1 and m_2 are stably equivalent, the relation $[m_1] = [m_2]$ follows immediately:

$$[m_1] = [m_1 \oplus n] - [n] = [m_2 \oplus n] - [n] = [m_2].$$

It remains to show that $[m_1] = [m_2]$ implies the existence of an $n \in M$ such that $m_1 \oplus n = m_2 \oplus n$. By construction of $\mathsf{K}^{\mathrm{Gro}}(M)$, there exist elements $a_1, \ldots, a_k, a'_1, \ldots a'_k, b_1, \ldots, b_l, b'_1, \ldots b'_l \in M$ for some $k, l \in \mathbb{N}$ such that in $\mathsf{F}(M)$ the following relation holds true:

$$m_1 - m_2 = \Big(\sum_{i=1}^{k} (a_i \oplus a'_i) - a_i - a'_i \Big) - \Big(\sum_{j=1}^{l} (b_i \oplus b'_i) - b_i - b'_i \Big).$$

This implies that in $\mathsf{F}(M)$, the following equation holds:

$$m_1 + \sum_{i=1}^{k}(a_i + a_i') + \sum_{j=1}^{l}(b_j \oplus b_j') = m_2 + \sum_{i=1}^{k}(a_i \oplus a_i') + \sum_{j=1}^{l}(b_j + b_j).$$

Since $\mathsf{F}(M)$ is free on elements of M, one concludes that the summands appearing on the left side of the equation are a permutation of the summands appearing on the right side. Hence

$$m_1 \oplus \bigoplus_{i=1}^{k}(a_i \oplus a_i') \oplus \bigoplus_{j=1}^{l}(b_j \oplus b_j') = m_2 \oplus \bigoplus_{i=1}^{k}(a_i \oplus a_i') \oplus \bigoplus_{j=1}^{l}(b_j \oplus b_j).$$

Putting $n := \bigoplus_{i=1}^{k}(a_i \oplus a_i') \oplus \bigoplus_{j=1}^{l}(b_j \oplus b_j')$, one obtains $m_1 \oplus n = m_2 \oplus n$. This finishes the proof. $\square$

Let us come back to our original problem and assume that $\lambda(m_1, n_1) = \lambda(m_2, n_2)$. Then one concludes

$$[m_1] + [n_2] = [m_2] + [n_1],$$

hence by the lemma there exists $n \in M$ such that

$$m_1 + n_2 + n = m_2 + n_1 + n. \tag{1.4}$$

If one defines now $(m_1, n_1) \sim (m_2, n_2)$ for $m_1, m_2, n_1, n_2 \in M$ if there exists $n \in M$ such that Eq. (1.4) holds true, then the lemma implies that $\lambda(m_1, n_1) = \lambda(m_2, n_2)$ exactly when $(m_1, n_1) \sim (m_2, n_2)$.

Lemma 1.5. *The relation $\sim$ on $M \times M$ is a congruence relation. This means in particular that for all $m_1, m_2, n_1, n_2, a, b \in M$ such that $(m_1, n_1) \sim (m_2, n_2)$ the relation*

$$(m_1 \oplus a, n_1 \oplus b) \sim (m_2 \oplus a, n_2 \oplus b)$$

holds true.

Proof. Clearly, the relation $\sim$ is symmetric and reflexive. let us show that it is transitive. To this end, assume $(m_1, n_1) \sim (m_2, n_2)$ and $(m_2, n_2) \sim (m_3, n_3)$. Then there exist $n, n' \in M$ such that

$$m_1 \oplus n_2 \oplus n = m_2 \oplus n_1 \oplus n \quad \textsf{and} \quad m_2 \oplus n_3 \oplus n' = m_3 \oplus n_2 \oplus n'.$$

Adding the two equalities, one obtains

$$m_1 \oplus n_3 \oplus (m_2 \oplus n_2 \oplus n \oplus n') = m_3 \oplus n_1 \oplus (m_2 \oplus n_2 \oplus n \oplus n'),$$

which proves that $\sim$ is transitive. If $m_1 \oplus n_2 \oplus n = m_2 \oplus n_1 \oplus n$, then

$$m_1 \oplus a \oplus n_2 \oplus b \oplus n = m_2 \oplus a \oplus n_1 \oplus b \oplus n,$$

which entails that $\sim$ is even a congruence relation. $\square$

Proposition 1.6. *For every commutative monoid M, the quotient space $M \times M/\sim$ of equivalence classes of the congruence relation $\sim$ is an abelian group which is canonically isomorphic to $\mathsf{K}^{\mathrm{Gro}}(M)$.*

Proof. Since $\sim$ is a congruence relation, $M \times M/\sim$ inherits from $M \times M$ the structure of an abelian monoid. Moreover, since $(m,n) \oplus (n,m) \sim (0,0)$, every element of $M \times M/\sim$ has an inverse, thus $M \times M/\sim$ is an abelian group. Since $\lambda(m_1,n_1) = \lambda(m_2,n_2)$ if and only if $(m_1,n_1) \sim (m_2,n_2)$ and since λ is surjective, it follows immediately that the quotient map $\overline{\lambda} : (M \times M/\sim) \to \mathsf{K}^{\mathrm{Gro}}(M)$ is well-defined and an isomorphism. □

Functorial properties

Sofar, we have defined $\mathsf{K}^{\mathrm{Gro}}$ only on objects of the category of abelian monoids. Let us now extend $\mathsf{K}^{\mathrm{Gro}}$ to a functor $\mathsf{K}^{\mathrm{Gro}} : \mathsf{AMon} \to \mathsf{AGrp}$. Assume to be given two abelian monoids M, N and a morphism of monoids $f : M \to N$. By the universal property of the Grothendieck group $\mathsf{K}^{\mathrm{Gro}}(M)$ there exists a uniquely determined group homomorphism, which we denote $\mathsf{K}^{\mathrm{Gro}}(f)$, such that the following diagram commutes.

$$
\begin{tikzcd}
M \arrow[r, "\kappa_M^{\mathrm{Gro}}"] \arrow[d, "f"'] & \mathsf{K}^{\mathrm{Gro}}(M) \arrow[d, "\mathsf{K}^{\mathrm{Gro}}(f)"] \\
N \arrow[r, "\kappa_N^{\mathrm{Gro}}"'] & \mathsf{K}^{\mathrm{Gro}}(N)
\end{tikzcd}
\tag{1.5}
$$

This in particular entails that

$$\mathsf{K}^{\mathrm{Gro}}(\mathrm{id}_M) = \mathrm{id}_{\mathsf{K}^{\mathrm{Gro}}(M)} \quad \text{and} \quad \mathsf{K}^{\mathrm{Gro}}(f_2 \circ f_1) = \mathsf{K}^{\mathrm{Gro}}(f_2) \circ \mathsf{K}^{\mathrm{Gro}}(f_1)$$

for abelian monoids M, M_1, M_2, M_3 and morphisms $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$. Hence $\mathsf{K}^{\mathrm{Gro}}$ is a functor from the category of abelian monoids to the category of abelian groups, indeed. One sometimes calls this functor the Grothendieck K-functor.

Theorem 1.7. *The Grothendieck-functor $\mathsf{K}^{\mathrm{Gro}} : \mathsf{AMon} \to \mathsf{AGrp}$ is left adjoint to the forgetful functor $\iota : \mathsf{AGrp} \to \mathsf{AMon}$.*

Proof. Let M be an abelian monoid, A an abelian group, and consider the map

$$\big(\kappa_M^{\mathrm{Gro}}\big)^* : \mathsf{AGrp}\big(\mathsf{K}^{\mathrm{Gro}}(M), A\big) \to \mathsf{AMon}\big(M, \iota(A)\big), \quad f \mapsto f \circ \kappa_M^{\mathrm{Gro}} .$$

By Diagram (1.5), this map is natural in M. Naturality in A is obvious by definition. Moreover, since $\mathsf{K}^{\mathrm{Gro}}$ satisfies the universal property **(Gro)** in Definition 1.1, $\big(\kappa_M^{\mathrm{Gro}}\big)^*$ is even bijective. The claim follows. □

Basic examples

Remark 1.8. Sometimes it happens that a set M carries two binary operations $\oplus$ and $\otimes$ which both induce on M the structure of an abelian monoid. To distinguish the corresponding two, possibly different, Grothendieck groups we denote them in such a situation by $\mathsf{K}^{\mathrm{Gro}}(M,\oplus)$ and $\mathsf{K}^{\mathrm{Gro}}(M,\otimes)$, respectively.

Example 1.9.

1. Consider the abelian monoid of natural numbers $(\mathbb{N},+)$ with addition as binary operation. Then $\mathsf{K}^{\mathrm{Gro}}(\mathbb{N},+) = (\mathbb{Z},+)$. On the other hand, one has $\mathsf{K}^{\mathrm{Gro}}(\mathbb{N},\cdot) = \{0\}$, but $\mathsf{K}^{\mathrm{Gro}}(\mathbb{N}^*,\cdot) = (\mathbb{Q}_{>0},\cdot)$.

2. If A is an abelian group, then by the universal property of the Grothendieck group one immediately obtains $\mathsf{K}^{\mathrm{Gro}}(A) = A$.

3. Consider the set of non-zero integres $\mathbb{Z}^*$ with multiplication $\cdot$ as binary operation. Then $\mathsf{K}^{\mathrm{Gro}}(\mathbb{Z}^*,\cdot) = (\mathbb{Q},\cdot)$.

4. Let X be a compact topological space, and $\mathsf{Vec}_{\mathbb{C}}(X)$ the category of complex vector bundles over X. Since every complex vector bundle over X is isomorphic to a subbundle of some trivial bundle $X \times \mathbb{C}^n$, the category of isomorphism classes of complex vector budnles over X is small. Denote by $\mathsf{Iso}(\mathsf{Vec}_{\mathbb{C}}(X))$ its set of objects. Then the direct sum of vector bundles over X induces the structure of an abelian monoid on $\mathsf{Iso}(\mathsf{Vec}_{\mathbb{C}}(X))$. The isomorphism class of the trivial vector bundle $X \times \{0\}$ of fiber dimension 0 serves as the zero element in $\mathsf{Iso}(\mathsf{Vec}_{\mathbb{C}}(X))$. The K-theory of the space X (in degree 0) is now defined as the Grothendieck-group of $\mathsf{Iso}(\mathsf{Vec}_{\mathbb{C}}(X))$ that means as the abelian group
$$\mathsf{K}^0(X) := \mathsf{K}^{\mathrm{Gro}}\big(\mathsf{Iso}(\mathsf{Vec}_{\mathbb{C}}(X))\big).$$
For further reading on the K-theory of compact topological spaces see [Ati89, Kar08].

2 The functor K_0 for a unital ring

Definition and fundamental properties

Let R be a unital (but possibly noncommutative) ring, and $R\text{-}\mathsf{Mod}_{\mathsf{fp}}$ the category of finitely generated projective left modules over R.

Proposition 2.1. *The category of isomorphism classes of finitely generated projective left R-modules is small. Denote by $\mathsf{Iso}(R\text{-}\mathsf{Mod}_{\mathsf{fp}})$ the set of isomorphism classes of finitely generated projective left R-modules. Then the direct sum in the abelian category $R\text{-}\mathsf{Mod}$ induces on $\mathsf{Iso}(R\text{-}\mathsf{Mod}_{\mathsf{fp}})$ the structure of an abelian monoid.*

Proof. Every finitely generated projective left R-module is isomorphic to a direct summand of some R^n, $n \in \mathbb{N}$, and the finitely generated projective left R-modules are characterized by this property. From this, it follows immediately that $\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})$ is small. To check the second claim, let $f : M_1 \to M_2$ and $g : N_1 \to N_2$ be two isomorphisms in $R\text{ -}\mathsf{Mod}_{\mathsf{fp}}$. Then $(f, g) : M_1 \oplus N_1 \to M_2 \oplus N_2$ is an isomorphism as well, hence $\oplus$ descends to a binary operation on $\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})$ which we will denote by the same symbol:

$$\oplus : \mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}}) \times$$

Iso(R -Mod$_{fp}$) $\to \mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})$ .It is immediate to prove that $\oplus$ is associative and commutative on $\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})$, and that the equivalence class of the zero module serves as neutral element. This proves the proposition. □

Definition 2.2. For every unital ring R one defines $\mathsf{K}_0(R)$, the K-theory of order 0 of R, by

$$\mathsf{K}_0(R) := \mathsf{K}^{\mathrm{Gro}}\big(\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})\big)\ .$$

Proposition 2.3. *Two finitely generated projective left R-modules M and N represent the same element in $\mathsf{K}_0(R)$ if and only if $M \oplus R^n \cong N \oplus R^n$ for some $n \in \mathbb{N}$.*

Proof. Clearly, if $M \oplus R^n \cong N \oplus R^n$ and $[M]$, $[N]$ denote the equivalence classes of M respectively N in $\mathsf{K}_0(R)$, then the equation

$$[M] = [M \oplus R^n] - [R^n] = [N \oplus R^n] - [R^n] = [N]$$

follows immediately. It remains to show the converse. Assume that $[M] = [N]$. Then, by definition of $\mathsf{K}^{\mathrm{Gro}}\big(\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})\big)$ there exist finitely generated projective left R-modules A_i, A'_i, B_i, B'_i, $i = 1, \ldots, k$ such that in $\mathsf{F}\big(\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})\big)$, the free abelian group over the set of isomorphism classes of finitely generated projective left R-modules, the equality

$$\overline{M} - \overline{N} = \sum_{i=1}^{k}(\overline{A_i \oplus A'_i} - \overline{A_i} - \overline{A'_i}) - \sum_{i=1}^{k}(\overline{B_i \oplus B'_i} - \overline{B_i} - \overline{B'_i})$$

holds true, where we have denoted by $\overline{M}$ the image of M in $\mathsf{F}\big(\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})\big)$ and likewise for the other left R-modules. This implies that

$$\overline{M} + \sum_{i=1}^{k}(\overline{B_i \oplus B'_i}) + \sum_{i=1}^{k}(\overline{A_i} + \overline{A'_i}) = \overline{N} + \sum_{i=1}^{k}(\overline{A_i \oplus A'_i}) + \sum_{i=1}^{k}(\overline{B_i} + \overline{B'_i})$$

which means that the R-modules appearing as summands on the left hand side are permutations of the summands appearing on the right hand side. Thus, in $\mathsf{Iso}(R\text{ -}\mathsf{Mod}_{\mathsf{fp}})$, the following equality holds true.

$$M \oplus \bigoplus_{i=1}^{k}(B_i \oplus B'_i) \oplus \bigoplus_{i=1}^{k}(A_i \oplus A'_i) = N \oplus \bigoplus_{i=1}^{k}(A_i \oplus A'_i) \oplus \bigoplus_{i=1}^{k}(B_i \oplus B'_i)$$

Hence we obtain $M \oplus P \cong N \oplus P$ for

$$P := \bigoplus_{i=1}^{k} (A_i \oplus A_i') \oplus \bigoplus_{i=1}^{k} (B_i \oplus B_i') \, .$$

Since P is a finitely generated projective left R-module, there exists a left R-module Q such that $P \oplus Q \cong R^n$ for some $n \in \mathbb{N}$. This entails

$$M \oplus R^n \cong M \oplus P \oplus Q \cong N \oplus P \oplus Q \cong N \oplus R^n \, ,$$

and the claim follows. □

Remark 2.4.

1. Note that since a finitely projective R-module is a direct sum of some R^n, the relation $M \oplus R^n \cong N \oplus R^n$ holds true, if and only if M and N are stably equivalent in the sense of Definition 1.4. This observation also shows that Proposition 2.3 is a direct consequence of Lemma 1.3.
2. Sometimes, one writes $\mathsf{K}_0^{\mathrm{alg}}(R)$ instead of $\mathsf{K}_0(R)$ to emphasize that one considers the algebraic K-theory of the ring R and not a topological version of K-theory. Note, however, that for a Banach-algebra A the topological K-theory of A in degree 0 coincides with its algebraic K-theory as defined above. This means in particular, that in this case the not so precise notation $\mathsf{K}_0(A)$ will not lead to any confusion.

Basic examples

Example 2.5.

1. Let $\mathbb{k}$ be a field. A finitely generated projective module over $\mathbb{k}$ is a $\mathbb{k}$-vector space of finite dimension. The isomorphism classes of finitely generated projective $\mathbb{k}$-modules are therefore uniquely determined by dimension. Moreover, under this characterization, the isomorphism class of the direct sum of two finitely generated projective $\mathbb{k}$-modules corresponds to the sum of the dimensions of the two modules. Hence, by Example 1.9.1 it follows that

$$\mathsf{K}_0(\mathbb{k}) \cong \mathsf{K}^{\mathrm{Gro}}(\mathbb{N}) = \mathbb{Z}.$$

2. Let X be a compact topological space. Recall that by the Serre–Swan Theorem the category $\mathsf{Vec}_{\mathbb{C}}(X)$ of complex vector bundles over X is equivalent to the category of finitely generated projective modules over the algebra $\mathcal{C}(X)$ of continuous functions on X, hence one has a natural isomorphism of monoids

$$\mathsf{Iso}(\mathsf{Vec}_{\mathbb{C}}(X)) \cong \mathsf{Iso}(\mathcal{C}(X) \text{ -}\mathsf{Mod}_{\mathsf{fp}}).$$

By Example 1.9.4 the K-theory of $\mathcal{C}(X)$ then has to coincide with the K-theory of the space X (in degree 0):

$$\mathsf{K}_0(\mathcal{C}(X)) \cong \mathsf{K}^0(X).$$

Note that, if X is a smooth manifold, one even has

$$\mathsf{K}_0(\mathcal{C}^\infty(X)) \cong \mathsf{K}^0(X),$$

where $\mathcal{C}^\infty(X)$ denotes the algebra of smooth functions on X.

3 The functor $\mathsf{K}_1^{\mathrm{alg}}$ for a unital ring

Prerequisites

Groups of invertible infinite matrices. Let R be a unital ring, and $n \in \mathbb{N}^*$. Recall that by $\mathrm{GL}_n(R) \subset \mathfrak{M}_{n\times n}(R)$ one denotes the group of invertible $n \times n$-matrices with entries in R. For natural $n \geq m > 0$ one has a natural embedding $\iota_{nm} : \mathrm{GL}_m(R) \to \mathrm{GL}_n$ which is defined by the requirement that $r = (r_{ij})_{1\leq i,j\leq m} \in \mathrm{GL}_m(R)$ is mapped to the matrix $\iota_{nm}(r) \in \mathrm{GL}_n(R)$ with entries

$$\big(\iota_{nm}(r)\big)_{ij} := \begin{cases} r_{ij}, & \text{if } 1 \leq i,j \leq m, \\ 1, & \text{if } i = j \text{ and } m < i \leq n, \\ 0, & \text{if } i \neq j \text{ and } i > m \text{ or } j > m. \end{cases}$$

By definition, $\big((\mathrm{GL}_n(R))_{n\in\mathbb{N}^*}, (\iota_{nm})_{m\leq n}\big)$ then forms a direct system of groups. It has a direct limit which is denoted by $\mathrm{GL}_\infty(R)$ and which can be represented as the set of all matrices $r = (r_{ij})_{i,j\in\mathbb{N}}$ with entries $r_{ij} \in R$ for which there is an $n \in \mathbb{N}$ such that

$$(3.1)\quad (r_{ij})_{1\leq i,j\leq n} \in \mathrm{GL}_n(R) \quad \text{and} \quad r_{ij} = \begin{cases} 1, & \text{if } i,j > n \text{ and } i = j, \\ 0, & \text{if } i > n \text{ or } j > n \text{ and } i \neq j. \end{cases}$$

The product of two elements $r, \tilde{r} \in \mathrm{GL}_\infty(R)$ is given by

$$r \cdot \tilde{r} := s, \quad \text{where} \quad s_{ij} := \sum_{k\in\mathbb{N}} r_{il} \cdot \tilde{r}_{kj} \ .$$

It is immediate to check that $r \cdot \tilde{r}$ is an element of $\mathrm{GL}_\infty(R)$. The unit element in $\mathrm{GL}_\infty(R)$ is given by the matrix e with components

$$e_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The set $\mathrm{GL}_\infty(R)$ of matrices $(r_{ij})_{i,j\in\mathbb{N}}$ satisfying (3.1) together with the product $\cdot$ froms a group indeed. Moreover, for every $n \in \mathbb{N}^*$ there is a natural embedding $\iota_n : \mathrm{GL}_n(R) \to \mathrm{GL}_\infty(R)$ which is defined by the requirement that $r = (r_{ij})_{1\leq i,j\leq n} \in \mathrm{GL}_n(R)$ is mapped to the matrix $\iota_n(r) \in \mathrm{GL}_\infty(R)$ with entries

$$\big(\iota_n(r)\big)_{ij} := \begin{cases} r_{ij}, & \text{if } 1 \leq i,j \leq n, \\ 1, & \text{if } i = j \text{ and } n < i, \\ 0, & \text{if } i \neq j \text{ and } i > n \text{ or } j > n. \end{cases}$$

It is straightforward to prove that $\big(\mathrm{GL}_\infty(R), (\iota_n)_{n\in\mathbb{N}^*}\big)$ is the direct limit of $\big((\mathrm{GL}_n(R))_{n\in\mathbb{N}^*}, (\iota_{nm})_{m\leq n}\big)$ indeed. Sometimes, one calls $\mathrm{GL}_\infty(R)$ the *group of invertible infinite matrices over R.*

Groups of elementary matrics. Let us now recall the definition and basic properties of the group of elementary matrices. To this denote for $\lambda \in R$, $n \in \mathbb{N}^* \cup \{\infty\}$ and all integers $i \neq j$ with $1 \leq i,j < n+1$ by $e^n_{ij}(\lambda)$ the matrix in $\mathrm{GL}_n(R)$ having entry λ at the i-th row and j-th column, entry 1 at all diagonal elements, and 0 at all other places. In other words, this means

$$\big(e^n_{ij}(\lambda)\big)_{kl} := \begin{cases} 1, & \text{if } k = l \text{ and } 1 \leq k < n+1, \\ \lambda, & \text{if } k = i \text{ and } l = j, \\ 0, & \text{if } k \neq l, \ (k,l) \neq (i,j), \text{ and } 1 \leq k,l < n+1. \end{cases}$$

A matrix of the form $e^n_{ij}(\lambda)$ is called an *elementary matrix over R of order n.* The subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices over R of order n is called the *group of elementary matrix over R of order n* and is denoted by $\mathrm{E}_n(R)$. By slight abuse of language one sometimes calls $\mathrm{E}_\infty(R)$ the *group of elementary matrices over R.*

It is immediate to check that under the group homomorphism ι_{nm} from above with $0 < m \leq n < \infty$, the group $\mathrm{E}_m(R)$ is mapped into $\mathrm{E}_n(R)$, and that $\big((\mathrm{E}_n(R))_{n\in\mathbb{N}^*}, (\iota_{nm})_{m\leq n}\big)$ is a direct system of groups. The direct limit of this direct systems is $\mathrm{E}_\infty(R)$ as one easily checks.

Recall that for two elements g,h of a group G one denotes by $[g,h]$ the commutator $ghg^{-1}h^{-1}$. With this notation, the following holds true.

Proposition 3.1. *The elementary matrics $e^\infty_{ij}(\lambda)$, $e^\infty_{ij}(\mu)$ satisfy for all $\lambda,\mu \in R$ the following relations.*

(i) $e^\infty_{ij}(\lambda) \cdot e^\infty_{ij}(\mu) = e^\infty_{ij}(\lambda+\mu)$, *if $i \neq j$,*

(ii) $\big[e^\infty_{ij}(\lambda), e^\infty_{kl}(\mu)\big] = 1$, *if $i \neq j$, $k \neq l$, $j \neq k$, and $i \neq l$,*

(iii) $\big[e^\infty_{ij}(\lambda), e^\infty_{jl}(\mu)\big] = e^\infty_{il}(\lambda \cdot \mu)$, *if $i \neq j$, $i \neq l$, and $j \neq l$,*

(iv) $\big[e^\infty_{ij}(\lambda), e^\infty_{ki}(\mu)\big] = e^\infty_{kj}(-\mu \cdot \lambda)$, *if $i \neq j$, $i \neq k$, and $j \neq k$.*

The essential tool for the proof of the proposition is the following result.

Lemma 3.2. *For $1 \leq i,j,k,l \leq n$ with $i \neq j$, $k \neq l$ and $i \neq l$ or $j \neq k$ one has*

$$\begin{aligned} e^n_{ij}(\lambda) \cdot e^n_{kl}(\mu) = {} & (1-\delta_{il})\, e^n_{il}\big(\lambda\mu\delta_{jk} + \lambda\delta_{jl} + \mu\delta_{ik}\big) + \\ & + (1-\delta_{jk})\delta_{il} 1 + \\ & + (1-\delta_{ik})\big(e^n_{kl}(\mu) - 1\big) + (1-\delta_{jl})\big(e^n_{ij}(\lambda) - 1\big), \end{aligned}$$

where δ_{rs} denotes the Kronecker symbol, i.e. $\delta_{rs} = 1$ for $r = s$ and $\delta_{rs} = 0$ for $r \neq s$.

Proof of the Lemma. Let us compute the components of the matrix $e^n_{ij}(\lambda)e^n_{kl}(\mu)$.

$$\left(e^n_{ij}(\lambda)\, e^n_{kl}(\mu)\right)_{rs} = \sum_t \left(e^n_{ij}(\lambda)\right)_{rt} \left(e^n_{kl}(\mu)\right)_{ts} =$$

$$= \begin{cases} \sum_t \delta_{rt}\delta_{ts} = \delta_{rs}, & \text{for } r \neq i,\ s \neq l, \\ \sum_t \delta_{rt}\delta_{ts} + \lambda\delta_{js} = \delta_{is}, & \text{for } r = i,\ s \neq l,\ j = l, \\ \sum_t \delta_{rt}\delta_{ts} + \lambda\delta_{js} = \delta_{is} + \lambda\delta_{js}, & \text{for } r = i,\ s \neq l,\ j \neq l, \\ \sum_t \delta_{rt}\delta_{ts} + \mu\delta_{rk} = \delta_{rl}, & \text{for } r \neq i,\ s = l,\ j = k, \\ \sum_t \delta_{rt}\delta_{ts} + \mu\delta_{rk} = \delta_{rl} + \mu\delta_{rk}, & \text{for } r \neq i,\ s = l,\ j \neq k, \\ \lambda\mu\delta_{jk} + \lambda\delta_{jl} + \mu\delta_{ik}, & \text{for } r = i,\ s = l,\ i \neq l, \\ 1, & \text{for } r = i,\ s = l,\ i = l,\ j \neq k. \end{cases}$$

The claim follows. ☐

Proof of the Proposition. ☐

Definition and fundamental properties of $\mathsf{K}_1^{\mathrm{alg}}$

Definition 3.3. Let R be a unital ring. Then $\mathsf{K}_1^{\mathrm{alg}}(R)$, the algebraic K-theory of degree 1 of R, is defined as the abelian group

$$\mathsf{K}_1^{\mathrm{alg}}(R) := \mathrm{GL}_\infty(R)/[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]\ .$$

Proposition 3.4. *For every unital ring R the following equality holds true.*

$$\mathsf{K}_1^{\mathrm{alg}}(R) = \mathrm{GL}_\infty^{\mathrm{ab}}(R) = \mathrm{GL}_\infty(R)/[\mathsf{E}_\infty(R), \mathsf{E}_\infty(R)] = \mathrm{GL}_\infty(R)/\mathsf{E}_\infty(R)\ .$$

Proof. The claim is immediate by definition of $\mathsf{K}_1^{\mathrm{alg}}(R)$ and results from elementary matrix theory as stated in the prerequisites. ☐

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