In single variable calculus we studied scalar-valued functions defined from $\mathbb{R} \to \mathbb{R}$ and parametric curves in the case of $\mathbb{R} \to \mathbb{R}^2$ and $\mathbb{R} \to \mathbb{R}^3$. In the study of multivariate calculus we've begun to consider scalar-valued functions of two variables in the case $\mathbb{R}^2 \to \mathbb{R}$. Let us now try to think of all the possible functions we may come across in the study of real variable calculus.

Different Types of Functions:

Scalar-valued functions from $\mathbb{R}$ to $\mathbb{R}$:
For example consider $f : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$f(x) = \sqrt{x} + 2^{x^2}.$$

Parametric curves from $\mathbb{R}$ to $\mathbb{R}^m$:

- *Planar curves*
  For example consider $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle.$$

- *Space curves*
  For example consider $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^3$. defined by

$$\mathbf{r}(t) = \left\langle \cos(t), \sin(t), \frac{t}{10} \right\rangle.$$

Scalar-valued functions of multiple variables from $\mathbb{R}^n$ to $\mathbb{R}$:
For example consider $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by

$$f(x, y) = x^2 - xy^2.$$

Vector-valued functions of multiple variables from $\mathbb{R}^n$ to $\mathbb{R}^m$:
For example consider $\mathbf{f} : \mathbb{R}^3 \mapsto \mathbb{R}^2$ defined by

$$\mathbf{f}(x, y, z) = \langle 3yz, 4x + y \rangle.$$

With all these new functions, we return to a familiar question:

How do we differentiate these things?

We have considered derivatives for differentiable functions from $\mathbb{R} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{R}^n$, and $\mathbb{R}^m \to \mathbb{R}$. Recognize that these are specific cases of functions from $\mathbb{R}^m \to \mathbb{R}^n$.

Derivatives of Diffent Types of Functions:

Scalar-valued functions from $\mathbb{R}$ to $\mathbb{R}$:
We define the derivative of a differentiable scalar function $f : \mathbb{R} \mapsto \mathbb{R}$ as

f' ______________________________.

Parametric curves from $\mathbb{R}$ to $\mathbb{R}^m$:
We define the derivative of any vector-valued function of one variable $f : \mathbb{R} \mapsto \mathbb{R}^n$, for $\mathbf{f}(t) = \langle x_1(t), \ldots, x_n(t) \rangle$ as

$\mathbf{f}'(t) = \langle x_1'(t), \ldots, x_n'(t) \rangle$ ______________

given each $x_i'(t)$ exists.

Scalar-valued functions of multiple variables from $\mathbb{R}^n$ to $\mathbb{R}$:
We define the derivative of a scalar-valued function $f : \mathbb{R}^n \mapsto \mathbb{R}$, given each partial of f exists and is continuous as

$\nabla f = \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle$ ______________.

We can also refer to this derivative as the ***gradient vector of*** f, and denote the gradient of ∇f.

Vector-valued functions of multiple variables from $\mathbb{R}^n$ to $\mathbb{R}^m$:
We define the derivative of a vector-valued function of two variables $\mathbf{f} : \mathbb{R}^3 \mapsto \mathbb{R}^2$, for $\mathbf{f}(x, y, z) = \langle u(x, y, z), v(x, y, z) \rangle$ as the 2×3 matrix

$$D\mathbf{f} = \begin{bmatrix} u_x(x, y, z) & u_y(x, y, z) & u_z(x, y, z) \\ v_x(x, y, z) & v_y(x, y, z) & v_z(x, y, z) \end{bmatrix} = \begin{bmatrix} \nabla u \\ \nabla v \end{bmatrix}$$

given each partial derivative exists and is continuous.

Compute the derivative for the following functions.

1. $\mathbf{f}(x,y) = \langle x^2 + y^2, xy \rangle$

   Solution: The function $\mathbf{f}$ maps from $\mathbb{R}^2 \mapsto \mathbb{R}^2$. This is a vector-valued function of multiple variables, and its derivative will result in a 2×2 matrix.

$$D\mathbf{f} = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}$$

2. $g(u,v) = 2u^2 - v^2$

   Solution: The function g maps from $\mathbb{R}^2 \mapsto \mathbb{R}$. This is a scalar-valued function of multiple variables, and its derivative will result in a gradient vector.

$$\nabla g(u,v) = \langle 4u, -2v \rangle$$

3. $\mathbf{r}(t) = \langle 3t^2, \ln|t|, 1, \cot(t) \rangle$

   Solution: The function $\mathbf{r}$ maps from $\mathbb{R}^2 \mapsto \mathbb{R}^2$. This is a parametric curve, and its derivative will result in a vector.

$$\mathbf{r}'(t) = \langle 6t, \tfrac{1}{t}, 0, -\sin(t) \rangle.$$

4. $\mathbf{w}(r,s,t) = \langle r^2 s, t^r + 3s^2 \rangle$

   Solution: The function $\mathbf{w}$ maps from $\mathbb{R}^3 \mapsto \mathbb{R}^2$. This is a vector-valued function of multiple variables, and its derivative will result in a 2×3 matrix.

$$D\mathbf{w} = \begin{bmatrix} 2rs & r^2 & 0 \\ t^r \ln|t| & 6s & rt^{r-1} \end{bmatrix}$$