This project reinforces our concept of composition of functions, parameterization, graphing, and differentiation, while gaining insight into why the idea of Lagrange multipliers works.

Our goal is to optimize the function $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ where $g(x, y) = x^2 + y^2 - 1$. We will solve this optimization problem using two different methods.

To begin, we graph the surface $f(x, y)$ below:
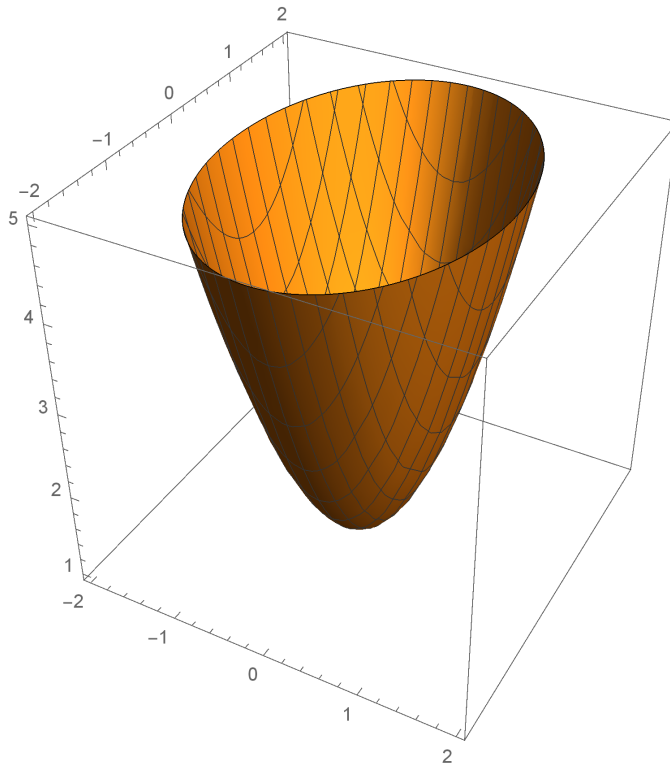
1. Draw the xyz-axes on the graph.

2. Draw the constraint $g(x, y) = x^2 + y^2 - 1 = 0$ on the xy-plane of the graph.

Notice we are finding extreme values of $f(x, y)$ over a curve, not over a 2D region.

Method 1
We can optimize $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ by first parameterizing the constraint equation.

3. Parameterize the constraint equation $g = 0$ with a planar curve $\mathbf{r}(t)$ in $\mathbb{R}^2$. Be sure to specify the domain for $\mathbf{r}(t)$.

4. Use Mathematica to plot of $f \circ \mathbf{r}$ which represents the function $f(x, y)$ subject to the constraint $g = 0$. Draw your output below. *Hint: Look up the ParametricPlot3D command on-line.*

5. Now using the fact that $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ are differentiable everywhere, write $h = f \circ \mathbf{r}$ just as a function of t, and then use the derivative of h to find the extreme values for the composition function.

We can examine the underlying details of Method 1 to find another method to optimize f where $g = 0$.

<u>Method 2</u>
Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ be arbitrary functions that are differentiable everywhere.

To find critical points of $f \circ \mathbf{r}$ we set $(f \circ \mathbf{r})' =$ _____. Given $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ are differentiable functions everywhere, we can use the Chain Rule to find

$$(f \circ \mathbf{r})' = \underline{\hspace{4cm}}.$$

Combining these two ideas we have

$$\underline{\hspace{3cm}} = \underline{\hspace{4cm}},$$

which shows the dot product of the vectors _____ and _____ is equal to zero. This means that these two vectors are _____.

Recall $\mathbf{r}(t)$ is a parameterization of $g = 0$, so $g(\mathbf{r}(t)) =$ _____. If we differentiate both sides we get

$$\underline{\hspace{4cm}} = \underline{\hspace{2cm}},$$

which shows the dot product of the vectors _____ and _____ is equal to zero. This means that these vectors are _____.

We now have $\mathbf{r}'(t)$ is orthogonal to both vectors _____ and _____. Since both of these vectors are orthogonal to $\mathbf{r}'(t)$ in $\mathbb{R}^2$, they must be scalar multiples of one another. Therefore, if λ is a constant, we can solve the system _____ $= \lambda$_____ to find all critical points of $f - \lambda g$.

This is called the method of *Lagrange Multipliers.*

Lagrange Multipliers

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are differentiable everywhere. To find the maximum and minimum values of f subject to the constraint $g = k$, where k is some constant [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the level set $g = k$]:

(a) Find all values of $\mathbf{x} = (x_1, x_2, \ldots x_n)$ and λ such that

$$\nabla f = \lambda \nabla g$$

and

$$g = k$$

(b) Evaluate f at all the points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ that result from step (a). The largest of these values is the maximum of f; the smallest is the minimum value of f subject to the constraint $g = k$.

6. Use the method of Lagrange multipliers to find all the extreme values of $f(x, y) = 2x^2 + y^2 + 1$ subject to the constraint $g = 0$ where $g(x, y) = x^2 + y^2 - 1$.