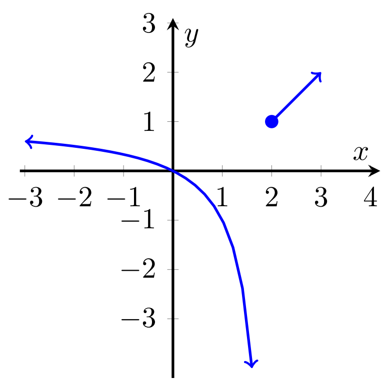
Part I: Review

In Calc I we learned about limits and continuity for functions $f : \mathbb{R} \mapsto \mathbb{R}$.

- Intuitively, $\lim_{x\to a} f(x) = L$ means that as x approaches a, $f(x)$ gets arbitrarily close to L.
- Remember that in order for this limit to exist, you must get the same limit as you approach a from **either** the left or the right.
- Recall also that $f(x)$ is **continuous** at $x = a$ if $\underline{\displaystyle\lim_{x\to a} f(x) = f(a)}$. Implicit in this definition are three requirements:
  - $f(a)$ exists
  - $\displaystyle\lim_{x\to a} f(x)$ exists.
  - The two numbers above are the same.
- Review problem: Determine if the following piecewise-defined function is continuous at $x = 2$ by taking limits. Graph the function.

$$f(x) = \begin{cases} \dfrac{x^2+2x}{x^2-4} & x < 2 \\ \dfrac{x^2-3x+2}{x-2} & x > 2 \\ 1 & x = 2 \end{cases}$$



Solution: First, by factoring and simplifying, we get

$$f(x) = \begin{cases} \dfrac{x}{x-2} & x < 2 \\ x-1 & x > 2 \\ 1 & x = 2 \end{cases}$$

Taking the limit from the right gives:

$\displaystyle\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-1) = 1$

Now taking the limit from the left gives:

$\displaystyle\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x}{x-2} = -\infty$

Since $\displaystyle\lim_{x\to 2^-} f(x)$ does not exist, this means $\displaystyle\lim_{x\to 2} f(x)$ does not exist. So $f(x)$ is not continuous at $x = 2$. However, we note that $\displaystyle\lim_{x\to 2^+} f(x) = f(2)$ (since they both equal 1), so $f(x)$ is right-continuous at $x = 2$.

Part II: Preview

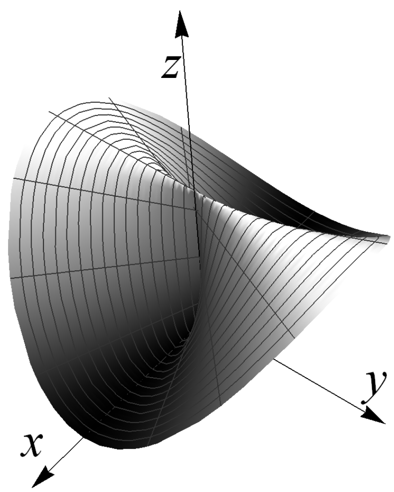
In Calc III, we'll now learn about limits of functions $f : \mathbb{R}^2 \mapsto \mathbb{R}$.

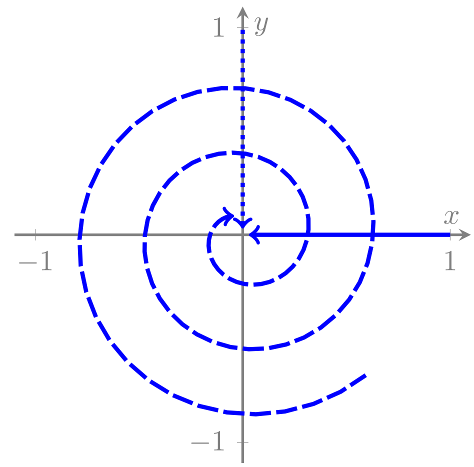
- Intuitively, $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$ means that as (x,y) approaches (a,b), $f(x,y)$ gets arbitrarily close to L.
- The domain is a plane, so now there are no longer just two directions to approach from. No matter which **path** we take as we approach (a,b), we must get the same limit. There are a **lot** of possible paths. This makes the problem harder. We will learn two techniques for answering the question of whether or not the limit exists.
- $f(x,y)$ is **continuous** at (a,b) if $\underline{\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)}$. Implicit in this definition are three requirements:
  - $f(a,b)$ exists.
  - $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ exists.
  - The two numbers above are the same.

Part III: Graphs

Look at your graph of the piecewise function in Part I on the previous page. Imagine walking along it towards $x = 2$ from the right. Notice that there is a sudden change in elevation; you will fall off the curve when you reach $x = 2$. Mathematically, this is because the limit from the left and the limit from the right are not equal.

With two input variables, the situation is more interesting, because there are far more than two ways to approach a point on a surface. Look at the function graphed on the left below.





1. Imagine walking on the surface towards $(x, y) = (0, 0)$, precisely along the y-axis. You will be walking at a constant altitude of $z = 1$. Draw this path with a dotted line on the graph on the above left. Then on the coordinate axes on the above right, draw a bird's-eye view of the path, using a dotted line.

2. Next imagine walking along the surface towards $(x, y) = (0, 0)$ but precisely along the x-axis. Draw the path with a solid line, both on the graph of the function and in the bird's-eye view on the right. What is your altitude along this path? Is it constant?

   Solution: I am walking at a constant altitude of 0. The ground around me is flat(ish).

3. Finally, walk along the surface in a spiral with your (x, y) position headed towards the origin. Draw the path with a dashed line, both on the graph of the function and in the bird's-eye view on the right. Now what is your walk along the surface like?

   Solution: I climb up and drop down increasingly steeply. It is dangerous! (Or it is fun, if you are a rock-climber.)

4. Do you think this function is continuous? Explain your intuition.

   Solution: The function is not continuous at $(x, y) = (0, 0)$, because the limit does not exist. I know that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist because the value of the limit differs depending on the path I take towards $(0, 0)$.

Part IV: A first technique - a way to prove a limit does not exist

Strategy for showing a limit does not exist:
If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then the limit must be the same, regardless of our path of approach to (a,b). Therefore, if we can find two paths of approach that produce different limits, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example: Consider the function $f(x,y) = \begin{cases} \dfrac{x^2+xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

1. Find the limit of $f(x,y)$ as we approach the point $(0,0)$ along the x-axis.

   Solution: Along the x-axis, we know that $y=0$. So the limit becomes $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \dfrac{x^2+0}{x^2+0} = \lim_{x\to 0} 1 = 1$.

2. Find the limit of $f(x,y)$ as we approach the point $(0,0)$ along the y-axis.

   Solution: Along the y-axis, we know that $x=0$. So the limit becomes $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \dfrac{0+0}{0+y^2} = \lim_{y\to 0} 0 = 0$.

3. Conclusion:

   Solution: Since the value of the limit is different along the two different paths, $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist, and thus $f(x,y)$ is not continuous at $(x,y)=(0,0)$.

Part V: A second technique - a way to prove a limit does exist

The difficulty in showing the limit of a function of two variables **does** exist is that we must show we get the same limit regardless of the path of approach. And there are countless paths of approach. If we are finding a limit as (x,y) approaches $(0,0)$, there is a sneaky trick. If we convert (x,y) to polar coordinates, then every path that approaches the origin has its radius r approaching 0.

Strategy for showing a limit does exist:
If we are trying to find $\lim_{(x,y)\to(0,0)} f(x,y)$, we convert $f(x,y)$ to polar coordinates and find $\lim_{r\to 0} f(r\cos\theta, r\sin\theta)$. If $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L$, regardless of how the values of θ vary with r, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$. The squeeze law is a valuable tool in this process.

Example: Consider the function $g(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ Is $g(x,y)$ continuous at $(x,y)=(0,0)$?

1. Convert the function to polar coordinates.

   Solution: Replacing x with $r\cos\theta$ and y with $r\sin\theta$ gives $g(x,y) = \dfrac{(r\cos\theta)^2(r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} = \dfrac{r^3\cos^2\theta\sin\theta}{r^2} = r\cos^2\theta\sin\theta$.

2. Use the squeeze law to show $\lim_{r\to 0} g(x,y) = 0$, regardless of how θ varies with r.

Solution: First notice that $-1 < \cos^2\theta\sin\theta < 1$. Thus $-r < r\cos^2\theta\sin\theta < r$. We know that $\lim_{r\to 0} -r = 0$ and $\lim_{r\to 0} r = 0$. By the squeeze law, $\lim_{r\to 0} r\cos^2\theta\sin\theta = 0$ as well. Thus, $\lim_{r\to 0} g(x,y) = 0$, regardless of how θ varies with r. So $\lim_{(x,y)\to(0,0)} g(x,y) = 0$.

3. Make a conclusion about the continuity of $g(x,y)$.

Solution: Since $g(0,0) = 0$ and $\lim_{(x,y)\to(0,0)} g(x,y) = 0$, g is continuous at $(0,0)$.

Some more practice problems: Determine the limit of each of the following functions as $(x,y)\to(0,0)$. Graphing in Mathematica may give you direction in what strategy to use.

1. $h(x,y) = \dfrac{xy}{x^2+y^2}$
2. $k(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$
3. $p(x,y) = \dfrac{x^2y}{x^4+y^2}$

Solution: For $h(x,y)$, first take the limit by approaching $(0,0)$ along the line $y = x$. This gives $\lim_{x\to 0} \dfrac{x^2}{x^2+x^2} = \lim_{x\to 0} \dfrac{x^2}{2x^2} = \dfrac{1}{2}$. Now take the limit by approaching $(0,0)$ along the line $y = -x$. This gives $\lim_{x\to 0} \dfrac{-x^2}{x^2+x^2} = \lim_{x\to 0} \dfrac{-x^2}{2x^2} = \dfrac{-1}{2}$. We found two different limits by approaching $(0,0)$ along two different paths, so $\lim_{(x,y)\to(0,0)} h(x,y)$ does not exist.

For $k(x,y)$ we will convert to polar coordinates. Replacing x with $r\cos\theta$ and y with $r\sin\theta$ gives $k(x,y) = \dfrac{r\cos\theta\cdot r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \dfrac{r^2\cos\theta\sin\theta}{r} = r\cos\theta\sin\theta$. Now notice that $-1 < \cos\theta\sin\theta < 1$. Thus $-r < r\cos\theta\sin\theta < r$. We know that $\lim_{r\to 0} -r = 0$ and $\lim_{r\to 0} r = 0$. By the squeeze law, $\lim_{r\to 0} r\cos\theta\sin\theta = 0$ as well. Thus, $\lim_{r\to 0} k(x,y) = 0$, regardless of how θ varies with r. So $\lim_{(x,y)\to(0,0)} k(x,y) = 0$.

The solution for $p(x,y)$, is left to you, with the hint to try taking limits along the curve $y = x^2$ and $y = -x^2$.