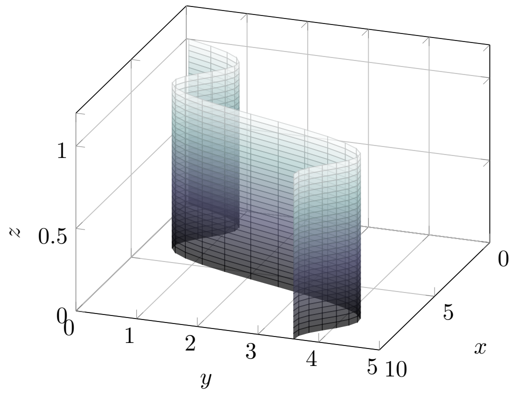
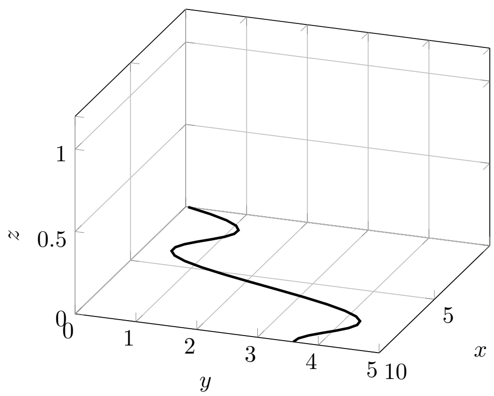
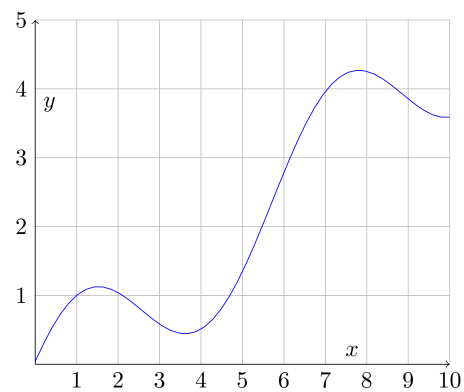
1. Suppose we want to calculate the area of the fence shown below.



Since the fence has a constant height of $h = 1$, the area (in square units) would be equal to $h = 1$ times the length of the curve shown on the left below. In the space given on the right, draw a "top-view" of the curve.

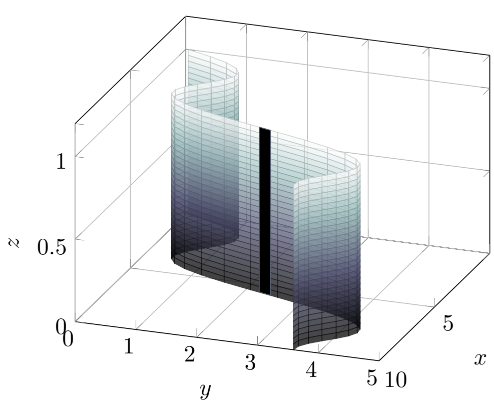




2. If the curve you've drawn above on the right is parameterized by $x(t)$, $y(t)$, for $0 \leq t \leq 10$, then write down an integral giving the arclength of the curve.

Solution: $\int_0^{10} \sqrt{[x'(t)]^2 + [y'(t))]^2}$

3. We can write the integral in another way by finding an expression for the area of the vertical "slats" of the fence, and summing those areas.



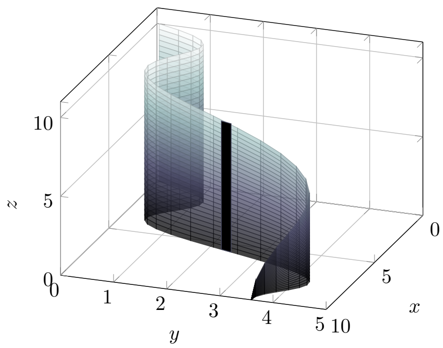
Area of shaded slat ≈

$$\underbrace{\left(\quad 1 \quad\right)}_{\text{height of slat}} \underbrace{\left(\quad \sqrt{[x'(t)]^2 + [y'(t)]^2}\Delta t \quad\right)}_{\underline{\text{times}}\text{ width of slat (a short diagonal segment)}}$$

Total area of fence =

$$\underbrace{\int_0^{10}}_{\text{Add}} \underbrace{\left(\quad 1 \quad\right)}_{\text{height of slat}} \underbrace{\left(\quad \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \quad\right)}_{\underline{\text{times}}\text{ width of slat (a short diagonal segment)}}$$

4. Now we'd like to find the area of the fence shown below, in which the height of the fence changes. Suppose that the base of the fence is again parameterized by $(x(t), y(t))$, for $0 \leq t \leq 10$, and the height of the fence is given by the function $f(x, y)$



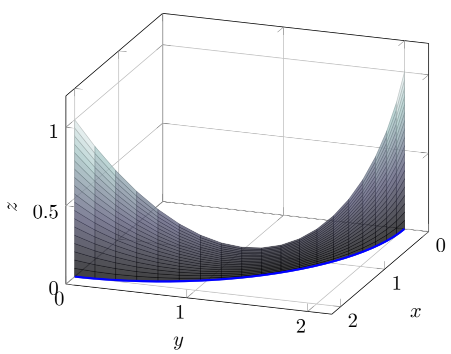
Area of shaded slat $\approx$

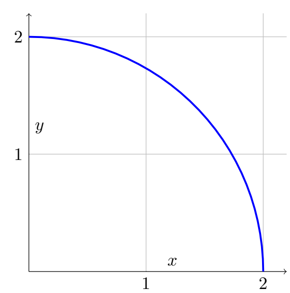
$$\underbrace{\Big(f(x(t), y(t))\Big)}_{\text{height of slat}} \; \underbrace{\Big(\quad \sqrt{[x'(t)]^2 + [y'(t)]^2}\Delta t \quad \Big)}_{\underline{\text{times}} \text{ width of slat (a short diagonal segment)}}$$

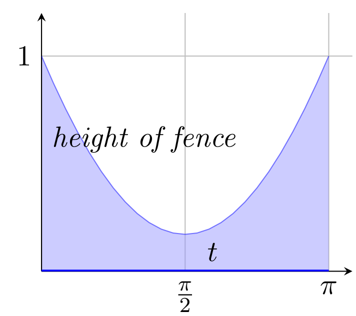
Total area of fence $=$

$$\underbrace{\int_0^{10}}_{\text{Add}} \underbrace{\Big(f(x(t), y(t))\Big)}_{\text{height of slat}} \; \underbrace{\Big(\quad \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \quad \Big)}_{\underline{\text{times}} \text{ width of slat (a short diagonal segment)}}$$

5. A fence is shown below on the left. The base of the fence runs along a quarter-circle of radius 2. The height of the fence is given by the function $f(x, y) = 3 - x - y$. The graph on the right shows the fence straightened so its base is a line. **(1)** In the middle graph below, draw a "top-view" of the fence. **(2)** Then on the other two graphs draw a thick line where this curves appears. **(3)** Next, find the arc length parameterization of the curve you drew, and use it to find the area of the fence. **(4)** Finally, find a formula for the graph on the right.







Solution: First we parameterize as follows:

$$\begin{cases} x(t) = 2\sin\left(\frac{t}{2}\right); & x'(t) = \cos\left(\frac{t}{2}\right) \\ y(t) = 2\cos\left(\frac{t}{2}\right); & y'(t) = -\sin\left(\frac{t}{2}\right) \end{cases}$$

$0 \leq t \leq \pi$

Substituting into the above formula gives

$\int_0^\pi (3 - 2\sin\left(\frac{t}{2}\right) - 2\sin\left(\frac{t}{2}\right))\sqrt{(\cos\left(\frac{t}{2}\right))^2 + (-\sin\left(\frac{t}{2}\right))^2}\, dt = \int_0^\pi 3 - 2\sin\left(\frac{t}{2}\right) - 2\cos\left(\frac{t}{2}\right) dt.$

Note that the integrand is exactly the formula for the function graphed on the right above.

Summary: The type of integral in these examples is called a **line integral** (though it might help you to think of it as a "curve integral" or a "path integral"). If C is a plane curve parameterized by $x(t)$ and $y(t)$ for $a \leq t \leq b$, then the **line integral of** f **along** C is

$$\int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

We often write this integral as $\displaystyle\int_C f(x, y)\, ds$. When you see it written this way, don't forget that it is often still useful to begin by parameterizing C. We will see line integrals again in Chapter 13.