Math 2400: Calculus III **Introduction to Surface Integrals - Generalizing the formula for surface area**

You recently learned how to find the area of a surface by parameterizing, then evaluating the appropriate integral. The first exercise is a review of that concept. In the second problem we will generalize the idea of surface area, introducing a new type of integral: surface integrals of scalar fields.

1. Find the surface area of the part of the surface $z^2 = 4x^2 + 4y^2$ lying between $z = 0$ and $z = 2$.

   (a) Find the intersection of the surface $z^2 = 4x^2 + 4y^2$ and $z = 2$.

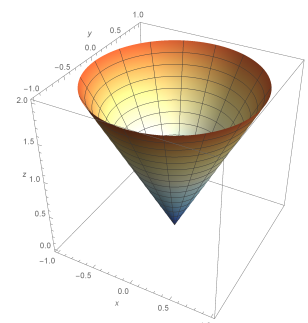
   (b) Graph the surface we are trying to find the area of.

   (c) Parameterize the surface using cylindrical coordinates to get $\mathbf{r}(r, \theta)$. Don't forget to include the intervals for the parameters.

   (d) A rectangle with dimensions Δr and $\Delta\theta$ gets mapped by the parameterization onto the surface, to a patch that is roughly a parallelogram. The area of this patch is approximately calculated by the formula $\Delta S = \underline{\quad |\mathbf{r}_r \times \mathbf{r}_\theta| \, \Delta r \, \Delta\theta \quad}$. Calculate the area ΔS for this surface.

   (e) Find the total surface area by integrating.

   Solution:

   The cone $z^2 = 4x^2 + 4y^2$ intersects the plane $z = 2$ when $2^2 = 4x^2 + 4y^2$, or $x^2 + y^2 = 1$. This is a circle of radius 1, at an altitude of $z = 2$. Since $z \geq 0$, we are dealing with only the upper nappe of the cone $z^2 = 4x^2 + 4y^2$. The x-z trace of the upper nappe of the cone $z^2 = 4x^2 + 4y^2$ has formula $z^2 = 4x^2$, or $z = 2|x|$. So it is narrower than a right-circular cone.

   To parameterize the surface using cylindrical coordinates, notice that the top view of the surface is a disc of radius 1. To fill this disc, our parameters range from $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Noting that for the upper nappe of the cone, $z = 2\sqrt{x^2 + y^2}$, our parameterization is given by $\mathbf{r}(r, \theta) = (r\cos\theta, r\sin\theta, 2r)$. Differentiating gives $\mathbf{r}_r = (\cos\theta, \sin\theta, 2)$ and $\mathbf{r}_\theta = (-r\sin\theta, r\cos\theta, 0)$. The area of a patch on the surface is roughly given by $\Delta S = |\mathbf{r}_r \times \mathbf{r}_\theta| \, \Delta r \, \Delta\theta = |(2r\cos\theta, 2r\sin\theta, r)| \, \Delta r \, \Delta\theta = \sqrt{5} r \Delta r \, \Delta\theta$.
   Integrating gives that the area of the cone is

   $$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \sqrt{5} r \, dr \, d\theta = 2\pi \int_0^1 \sqrt{5} r \, dr = \sqrt{5}\pi.$$

   Since the surface is a cone, we can confirm our result using the formula for the lateralsurface area of a cone, $S = \pi r s$, where s is the slant height. Here the radius is 1 and the slant height is $\sqrt{5}$, confirming our result.

2. In an integral for surface area, we use the differential $|\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$, which geometrically represents:

   Solution: the area of the tiny paralellogram which the parameterization maps a du-by-dv rectangle onto.

3. A student asks "We have calculated $\iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$, which is a double integral. I thought a double-integral gives a volume, so why does this integral only give an area?" Respond to this student.

   Solution: In the double integral $\iint_R f(x, y) \, dA$, the differential dA represents the area of a tiny rectangle. If $f(x, y)$ is a height, then $f(x, y) \, dA$ gives a volume of a rectangular box with a very tiny base. The integral then gives a volume. In our integral, all of the expression $|\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$ represents an area dA. Thus there is no height in the integrand. Our integral sums areas, not volumes, so our final result represents an area, not a volume.

We have seen that the area of a parameterized surface $\mathbf{r}(u,v)$ over the region R can be found by calculating

$$\iint_R 1\,|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv.$$

The idea of a surface integral is to generalize by replacing the "1" with an arbitrary function.

4. Suppose the surface of problem 1 has a variable density of $\rho(x,y,z) = \sqrt{4-z^2}$. Find its total mass. Assume the units of mass are grams, and the units of distance are meters.

   (a) What are the units of ρ?

   (b) In the previous problem, we calculated the area of a patch as $|\mathbf{r}_r \times \mathbf{r}_\theta|\,\Delta r\,\Delta\theta =$ $\underline{\quad\sqrt{5}r\Delta r\,\Delta\theta\quad}$.

   (c) What is the mass of a patch?

   (d) Integrate to find the total mass.

   Solution: The density ρ is a mass per unit area, so in this case it is given in grams per square meter. The mass of small patch on the surface is given by the density at that location times the area of the patch. This gives

   $$\Delta m = \sqrt{4-z^2}\sqrt{5}r\Delta r\Delta\theta.$$

   Along the surface of the cone, $z^2 = 4x^2+4y^2 = 4r^2$. Substituting gives

   $$\Delta m = \sqrt{5}\sqrt{4-4r^2}r\Delta r\Delta\theta.$$

   Using the same parameterization as in the previous problem, we calculate the total mass:

   $$\begin{aligned} m &= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\sqrt{5}\sqrt{4-4r^2}r\,dr\,d\theta \\ &= 2\sqrt{5}\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\sqrt{1-r^2}r\,dr\,d\theta \\ &= 4\sqrt{5}\pi\int_{r=0}^{1}\sqrt{1-r^2}r\,dr = \frac{4\sqrt{5}\pi}{3} \end{aligned}$$

   To do a plausibility-check on this result, recall that the area of the surface is $\sqrt{5}\pi$. This means that the average density of the surface must be $\frac{4}{3}$. The density of the cone decreases from $\rho = 2$ at the bottom point of the cone to $\rho = 0$ at the very top. The density at $z = 1$, halfway up the cone, is $\rho = sqrt3$, which is larger than what we calculated as the average density of the cone. This makes sense since more of the cone lies above that halfway point, and the upper portion is less dense than the lower portion.

> The surface integral of a function $f(x,y,z)$ (i.e., a scalar field) over a surface S is written $\iint_S f(x,y,z)\,dS$. It is computed by $\underline{\quad\text{parameterizing}\quad}$ the surface S as $\mathbf{r}(u,v)$ and computing $\underline{\quad\iint_R f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA\quad}$, (where R defines the domain of u and v).

5. Practice at home: Re-do the integrals in problems 1 and 4 using spherical coordinates. Make sure you get the same numerical answers as when you did the calculations in cylindrical coordinates.

6. Practice at home: Use spherical coordinates to find the area of the part of the sphere $x^2+y^2+z^2 = 16$ that lies above the plane $z = 2$. Then find the mass of that surface if the density is given by $\rho = z$.