Answer the following questions about the "Gradient graphically" project:

1. Confirm analytically that the gradient of each of formulas for the potential functions matches the formula for the gradient vector field.

2. The level curves for the cone (graph ♡) and the paraboloid (graph ♢) are both concentric circles. How did you determine which set of level curves match the cone?

   Solution: If I slice the cone with cuts parallel to the xy-plane at even intervals (for example, at $z = 1$, $z = 2$, $z = 3$, etc.), then the radius of the circles grow linearly. This is because the sides have a constant slope. This means that the spacing between the circles is consistent (as in level curves graph A.) On the other hand, with the paraboloid the circles grow more slowly as I move upwards (because the sides of the surface get steeper as I move upwards), so the concentric circles in the level curves becomes closer and closer together.

3. The gradient field for the cone and the paraboloid both consist of vectors that all point directly away from the origin. How did you determine which of those two fields match the cone?

   Solution: Recall that the gradient points in the direction of steepest ascent of the potential function. If we imagine sitting on the surface of each of the two potential functions, then it makes sense that we should walk in a direction away from the origin to climb most steeply. Further, the magnitude of the gradient vector is the slope of this steepest ascent. On the parboloid, we climb more steeply as we move further from the origin. Thus the gradient vectors become longer the further we are from the origin. On the cone, on the other hand, the steepness stays constant no matter how far we are from the origin, so the vectors in the gradient vector field should all be of the same length.

4. Look at each gradient vector field. Only by looking at this field, try to imagine the surface of the potential function it represents. Try the same thing by looking at the graphs of the level curves.

   Solution: If we imagine the vectors as indicating the direction of the steepest climb, and their length indicating the steepness, we can see the potential function popping out of the page. The same thing can be done with the level curves.

5. The gradient vector field on card 3 matches the level curves from Card C, but it looks like no matter how it is oriented, the vectors point perpendicular to the level curves. How do you determine which way to orient it? Do you require the graph of card ♣ do confirm this? Explain.

   Solution: The gradient vector points perpendicular to the level curve, and points in the direction of maximum **increase** of the potential function (in other words, it points "uphill"). Only by looking at the graph of the potential function can we confirm that the high points on the hyperbolic paraboloid lie along the x-axis, and thus that darker orange on the level curves corresponds to higher values. The arrows on the gradient vector field must point towards higher values, or darker orange. This tells us which way to orient the card.

6. Explain how to confirm directly that formula card b, matches potential function graph $\diamondsuit$.

   Solution: We can look first at the level curves, that is, the cross-sections perpendicular to the z-axis. This means setting z to a constant k. So all the level curves are of the form $k = x^2 + y^2$. These are circles centered at the origin, matching what we see when we slice the potential function parallel to the xy-plane. Now setting $x = 0$ to see the yz-trace, we have $z = y^2$, a parabola. This again matches what we see when we slice parallel to the yz-plane. Finally, the xz-trace is the parabola $z = x^2$, matching the graph of the paraboloid.

7. Explain how to directly confirm that formula card a matches the level curves graph C.

   Solution: To find the level curves, I look at cross-sections perpendicular to the z-axis. This means setting z to a constant k. So all the level curves are of the form $k = x^2 - y^2$. These are hyperbolas with vertices along the x-axis for positive values of k, and along the y-axis for negative value of k. Graph C of the level curves in the only one with hyperbolas.

8. Look at card γ, giving the formula for a vector field. Calculate the magnitude of the gradient vector, then explain how this confirms the match with vector field card 2. Also explain why the vector at each point in the plane is directed exactly away from the origin.

   Solution: The magnitude is $\sqrt{\dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2}} = 1$. Card 2 is the only one with vectors of constant magnitude. The vector field points in the direction $\langle x, y \rangle$, directly away from the origin.

9. Suppose a potential function is shifted upward in the z-direction. For example, $f(x, y) = x^2 + y^2$ becomes $f(x, y) = x^2 + y^2 + 5$. How does this affect the graphs of the level curves, and the gradient vector field?

   Solution: The graph of the level curves will be the same, though the levels will be labelled differently. The gradient vector field is unchanged.

10. Look at vector field 1 and potential function $\diamondsuit$. A student incorrectly says that these vectors point perpendicular to the surface of the potential function. Explain what is wrong with their thinking, and explain how to find the vectors that point perpendicular to the surface of the potential function itself.

    Solution: The gradient vector field is two-dimensional, so it just points in the xy direction of fastest increase. The vectors perpendicular to the surface are 3D vectors. The gradient vector points perpendicular to the level curves of the surface, not to the surface itself. The vector $\langle f_x, f_y, -1 \rangle$ points perpendicular to the surface of the function f.