Solutions to Math 2400 Midterm 2

March 11, 2019

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50 AM
☐	Section 002	Harrison Stalvey	8:00–8:50 AM
☐	Section 003	Daniel Martin	9:00–9:50 AM
☐	Section 004	Albert Bronstein	9:00–9:50 AM
☐	Section 005	Xingzhou Yang	10:00–10:50 AM
☐	Section 006	Mark Pullins	10:00–10:50 AM
☐	Section 007	János Englander	10:00–10:50 AM
☐	Section 008	John Willis	12:00–12:50 PM
☐	Section 009	Taylor Klotz	1:00–1:50 PM
☐	Section 010	János Englander	2:00–2:50 PM
☐	Section 011	Harrison Stalvey	2:00–2:50 PM
☐	Section 012	Xingzhou Yang	3:00–3:50 PM
☐	Section 013	Trevor Jack	4:00–4:50 PM

Question	Points	Score
1	8	
2	5	
3	5	
4	10	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
12	8	
13	8	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 95 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like 100/7 or expressions like $\ln(3)/2$ as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (8 points) **Note: No partial credit for this problem.**
   Let $f(x, y, z) = x^2 y + e^{2z}$. Compute the following:

   (a) $\dfrac{\partial f}{\partial x} = \underline{2xy}$.

   (b) $\dfrac{\partial^2 f}{\partial z^2} = \underline{4e^{2z}}$.

   (c) $\dfrac{\partial^2 f}{\partial x \partial y} - \dfrac{\partial^2 f}{\partial y \partial x} = \underline{0}$.

   (d) $\boldsymbol{\nabla} f(1, -1, 0) = \underline{\langle -2, 1, 2 \rangle}$.

2. (5 points) Circle the answer that best describes each statement.

(a) If f is defined at $(0,0)$ and $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, then f is continuous at $(0,0)$.

(A) Always true

(B) $\boxed{\text{Sometimes true}}$

(C) Never true

(b) If $f(x,y) \to L$ as $(x,y) \to (0,0)$ along every straight line through $(0,0)$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$.

(A) Always true

(B) $\boxed{\text{Sometimes true}}$

(C) Never true

(c) If $z = f(x,y)$ is a function of two variables and $\nabla f(a,b) = \langle 0,0\rangle$, then f has a local maximum or minimum at (a,b)

(A) Always true

(B) $\boxed{\text{Sometimes true}}$

(C) Never true

(d) If $f_x(x,y) = 0$ and $f_y(x,y) = 0$ for all (x,y), then f is constant.

(A) $\boxed{\text{Always true}}$

(B) Sometimes true

(C) Never true

(e) If $f(x,y)$ is a differentiable function and $\vec{u}$ is a unit vector, then the directional derivative $D_{\vec{u}}f(a,b)$ is parallel to $\vec{u}$.

(A) Always true

(B) Sometimes true

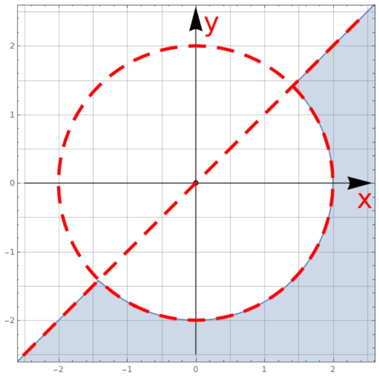
(C) $\boxed{\text{Never true}}$

3. (5 points) Let $f(x,y) = \dfrac{\ln(x-y)}{\sqrt{4-x^2-y^2}}$.

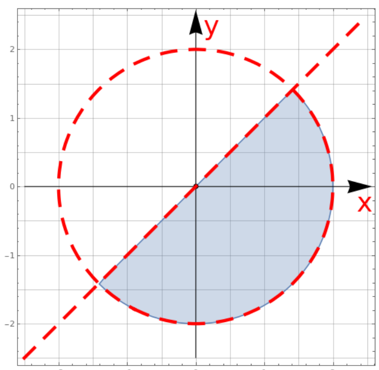
Its domain $= \{(x,y) | x - y > 0,\ 4 - x^2 - y^2 > 0\} = \{(x,y) | y < x,\ x^2 + y^2 < 4\}$.

Which one of the following shaded regions is the domain of f? **(B)**.

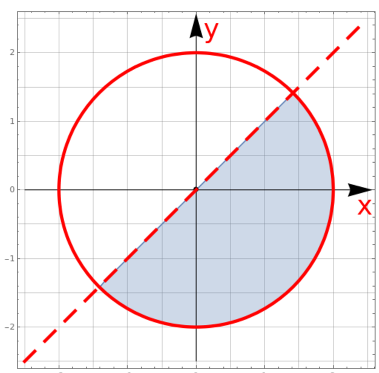
(A)



(B)



(C)



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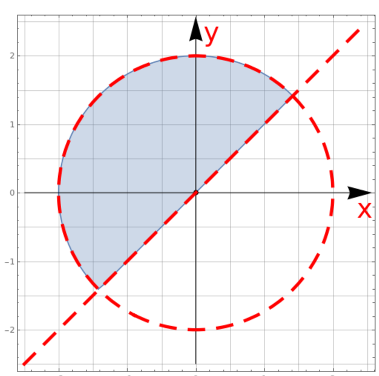
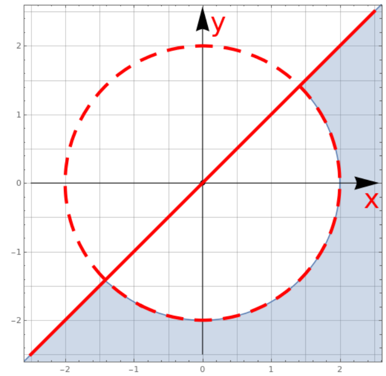
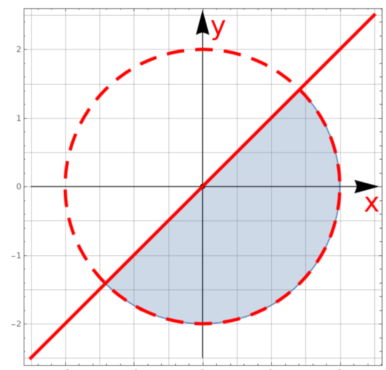


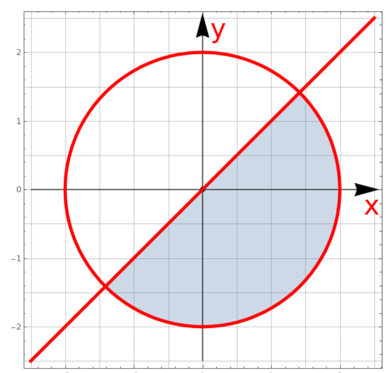
(E)



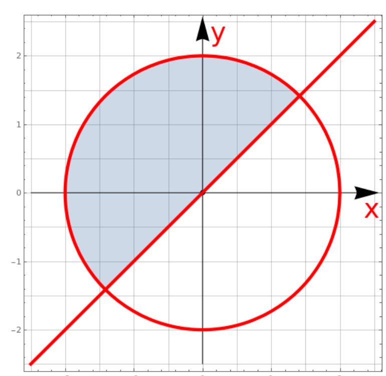
(F)



(G)

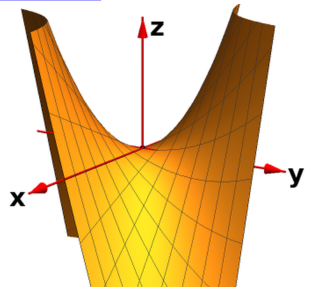


(H)

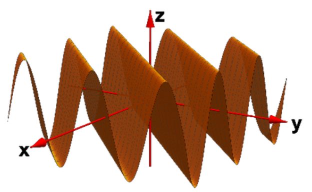


4. (10 points) Match each 3D surface with one of the contour plots, and one of the equations.

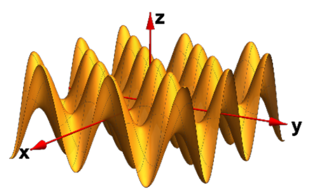
(a) **(4), (E)**



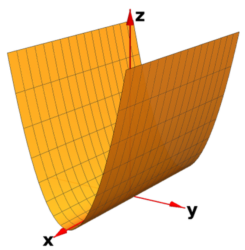
(b) **(5), (B)**



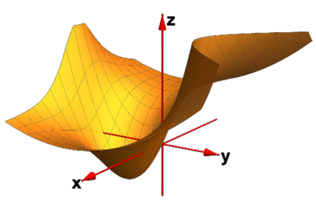
(c) **(2), (A)**



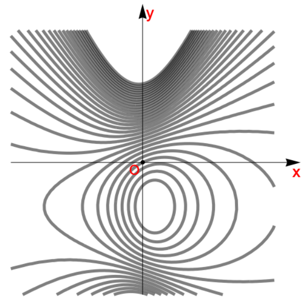
(d) **(3), (D)**



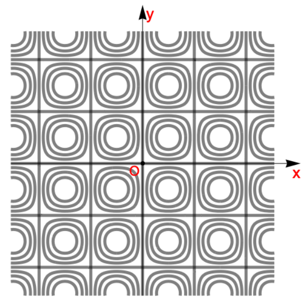
(e) **(1), (C)**



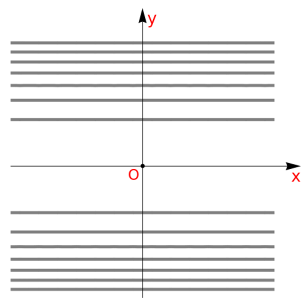
(1)



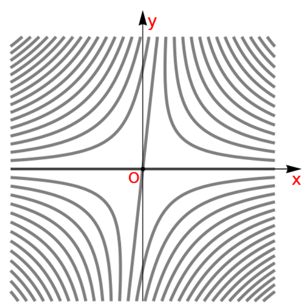
(2)



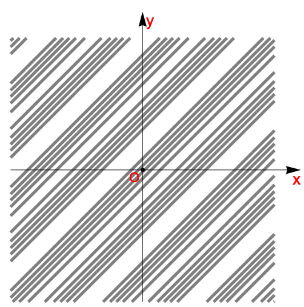
(3)



(4)



(5)



(A) $z = \sin(x)\sin(y)$

(B) $z = \sin(x - y)$

(C) $z = \dfrac{x^2 - x + y^2 + 2y}{x^2 + 1}$

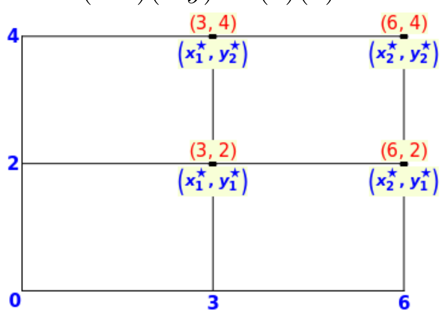
(D) $z = y^2$

(E) $z = y^2 - 9xy$

5. (8 points) Let S be the surface given by $z = xy$, and let R be the rectangle $[0, 6] \times [0, 4]$.

(a) Taking the sample points to be the **upper right corners**, use a **Riemann sum** with $m = 2$ and $n = 2$ to estimate the volume of the solid that lies below S and above R.

Solution: Let $f(x, y) = xy$. Since $m = 2$, $n = 2$, $\Delta x = \dfrac{6-0}{2} = 3$, $\Delta y = \dfrac{4-0}{2} = 2$, $\Delta A = (\Delta x)(\Delta y) = (3)(2) = 6$.



The sample points are the upper right corners.

(x_i^*, y_j^*)	(x_1^*, y_1^*)	(x_1^*, y_2^*)	(x_2^*, y_1^*)	(x_2^*, y_2^*)
sample pt	$(3, 2)$	$(3, 4)$	$(6, 2)$	$(6, 4)$

$$\begin{aligned}\text{volume} &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A = \Delta A \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i^*, y_j^*)\\ &= \Delta A\left[f(3,2) + f(3,4) + f(6,2) + f(6,4)\right]\\ &= \Delta A\left[(3)(2) + (3)(4) + (6)(2) + (6)(4)\right]\\ &= (6)\left[6 + 12 + 12 + 24\right] = \boxed{\mathbf{324}}\end{aligned}$$

(b) Calculate the **exact** volume of the solid that lies below S and above R.

Solution: By **Fubini**'s Theorem, the exact volume is

$$\text{volume} = \iint_R xy\,\mathrm{d}A = \int_0^6 \int_0^4 xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^6 \left.\frac{xy^2}{2}\right|_{y=0}^{y=4} \mathrm{d}x = \int_0^6 8x\,\mathrm{d}x = \left.4x^2\right|_{x=0}^{x=6}$$

$$= 4\left(6^2 - 0^2\right) = \boxed{\mathbf{144}}$$

$$\underset{=}{\text{or}} \int_0^4 \int_0^6 xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^4 \left.\frac{x^2 y}{2}\right|_{x=0}^{x=6} \mathrm{d}y = \int_0^4 18y\,\mathrm{d}y = \left.9y^2\right|_0^4$$

$$= 9(4^2 - 0) = \boxed{\mathbf{144}}$$

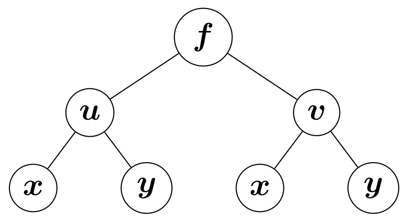
$$\underset{=}{\text{or}} \int_0^4 \int_0^6 xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^4 y\,\mathrm{d}y \int_0^6 x\,\mathrm{d}x = \left.\frac{y^2}{2}\right|_0^4 \left.\frac{x^2}{2}\right|_0^6 = \frac{4^2}{2}\cdot\frac{6^2}{2} = \boxed{\mathbf{144}}$$

6. (8 points) Let f be a differentiable function, $g(x, y) = f(u, v)$, where $u = x^2 - y^2$, $v = y^2 - x^3$.

$g(1,2) = 11$	$g(-3,3) = 7$	$f(1,2) = 20$	$f(-3,3) = 11$
$f_u(1,2) = 4$	$f_v(1,2) = 5$	$f_u(-3,3) = 2$	$f_v(-3,3) = -1$

Evaluate $g_x(1, 2)$ based on the values in the above table.

Solution: As $x = 1$, $y = 2$, $u = (1)^2 - (2)^2 = 1 - 4 = -3$, $v = (2)^2 - (1)^3 = 4 - 1 = 3$. By the **Chain Rule**,



$$\begin{aligned} g_x(x, y) &= f_u \frac{\partial u}{\partial x} + f_v \frac{\partial v}{\partial x} = f_u(u, v) \frac{\partial}{\partial x}\left(x^2 - y^2\right) + f_v(u, v) \frac{\partial}{\partial x}\left(y^2 - x^3\right) \\ &= 2x f_u(u, v) - 3x^2 f_v(u, v) \\ g_x(1, 2) &= 2(1) f_u(-3, 3) - 3(1)^2 f_v(-3, 3) \\ &= 2(2) - 3(-1) = 4 + 3 = \boxed{7} \end{aligned}$$

7. (8 points) A surface is represented by $z = f(x, y)$ with f differentiable. Let $\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. Suppose that $f_x(2,1) = 5$ and $D_{\vec{u}} f(2,1) = 3\sqrt{2}$.

(a) Find $f_y(2,1)$.

Solution: Note $\vec{u}$ is a unit vector. By the definition of directional derivative,

$$\begin{aligned} D_{\vec{u}} f(2,1) &= \nabla f(2,1) \cdot \vec{u} = \langle f_x(2,1),\ f_y(2,1) \rangle \cdot \vec{u} \\ &= \langle 5,\ f_y(2,1) \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{5}{\sqrt{2}} + \frac{1}{\sqrt{2}} f_y(2,1) \\ f_y(2,1) &= \sqrt{2} D_{\vec{u}} f(2,1) - 5 = \sqrt{2} \cdot 3\sqrt{2} - 5 = 6 - 5 = \boxed{\mathbf{1}} \end{aligned}$$

(b) Assume $f(2,1) = 3$. Use the result in (a) to find a vector in the xy-plane that is tangent to the level curve $f(x, y) = 3$ at the point $(2,1)$.

Solution: Note the tangential vector at $(2,1)$, say $\vec{v} = \langle v_1, v_2 \rangle$, is **orthogonal** to the gradient vector $\nabla f(2,1) = \langle f_x(2,1),\ f_y(2,1) \rangle = \langle 5, 1 \rangle$. So we have

$$\vec{v} \cdot \nabla f(2,1) = 0 \iff \langle v_1, v_2 \rangle \cdot \langle 5, 1 \rangle = 0 \iff 5v_1 + v_2 = 0 \iff v_2 = -5v_1$$

So $\vec{v} = \langle v_1, v_2 \rangle = \langle v_1, -5v_1 \rangle = \boxed{\boldsymbol{v_1 \langle 1, -5 \rangle}}$, where v_1 can be any non-zero real number.

8. (8 points) Find all critical points of the function

$$f(x,y) = \frac{2}{3}x^3 + 2x^2 + y^2 - 2xy$$

and classify each as a local maximum, local minimum, saddle point, or not enough information.

Solution: To find all the critical points, we solve $\nabla f(x,y) = \langle 0,0\rangle$, or $\langle f_x, f_y\rangle = \langle 0,0\rangle$. To classify them, we need to use the **Second Derivative Test**.

$$\begin{aligned}
f_x &= \frac{\partial}{\partial x}\left(\frac{2}{3}x^3 + 2x^2 + y^2 - 2xy\right) = 2x^2 + 4x - 2y \\
f_y &= \frac{\partial}{\partial y}\left(\frac{2}{3}x^3 + 2x^2 + y^2 - 2xy\right) = 2y - 2x \\
f_{xx} &= \frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x}\left(2x^2 + 4x - 2y\right) = 4x + 4 \\
f_{xy} &= \frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}\left(2x^2 + 4x - 2y\right) = -2 = f_{yx} \\
f_{yy} &= \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial y}(2y - 2x) = 2 \\
D &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 = (4x+4)(2) - (-2)^2 = 8x + 4
\end{aligned}$$

Solve

$$\langle f_x, f_y\rangle = \langle 0,0\rangle \iff \begin{cases} f_x = 2x^2 + 4x - 2y = 0 \\ f_y = 2y - 2x = 0 \end{cases} \iff \begin{cases} x^2 + 2x - y = 0 \text{ ①} \\ y = x \text{ ②} \end{cases}$$

By ②, we substitue x for y in ①, and we get

$$x^2 + x = 0 \iff x(x+1) = 0 \iff x = 0, \text{ or } x = -1$$

Using ② again we get two critical points $\boxed{\mathbf{(0,0),\ (-1,-1)}}$

At $(0,0)$, $D(0,0) = 8(0) + 4 = 4 > 0$, $f_{xx}(0,0) = 4(0) + 4 = 4 > 0$, so by the 2nd derivative test, $\boxed{\mathbf{(0,0)\ is\ a\ local\ minimum\ point}}$.

At $(-1,-1)$, $D(-1,-1) = 8(-1) + 4 = -4 < 0$, so by the 2nd derivative test, $\boxed{\mathbf{(-1,-1)\ is\ a\ saddle\ point}}$.

9. (8 points) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-3x^2y+y^2}{4x^2+4y^2}$ if it exists, and then prove it. Otherwise, explain why it does not exist.

Solution: Let $f(x,y) = \dfrac{x^2-3x^2y+y^2}{4x^2+4y^2}$. Use polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$. Then $x^2+y^2=r^2$.

$$f(x,y) = \frac{x^2-3x^2y+y^2}{4x^2+4y^2} = \frac{x^2+y^2-3x^2y}{4(x^2+y^2)} = \frac{r^2-3r^3\cos^2\theta\sin\theta}{4r^2} = \frac{1}{4} - 3r\cos^2\theta\sin\theta$$

Since for any θ, $|\sin\theta| \le 1$, $|\cos\theta| \le 1$, we have

$$0 \le \left|f(x,y) - \frac{1}{4}\right| = \left|-3r\cos^2\theta\sin\theta\right| = 3r|\cos\theta|^2|\sin\theta| \le 3r$$

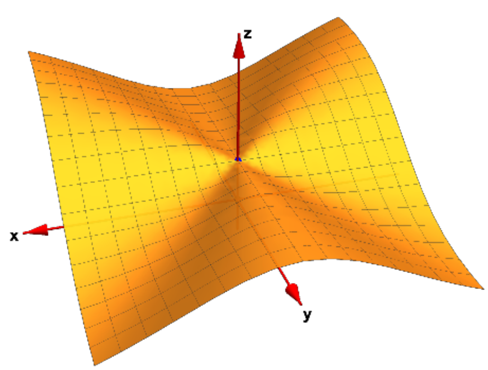
As $(x,y)\to(0,0)$, $r = \sqrt{x^2+y^2} \to 0$, and so the lower bound and upper bound of $\left|f(x,y)-\dfrac{1}{4}\right|$ both approach 0 as $(x,y)\to(0,0)$. By the **Squeeze Theorem**, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-3x^2y+y^2}{4x^2+4y^2} = \dfrac{1}{4}$. This proves the existence of the limit.

Solution 2: Let $f(x,y) = \dfrac{x^2-3x^2y+y^2}{4x^2+4y^2}$. Pick a path, for example, $C: y=0$. Then $f(x,y) = \dfrac{x^2}{4x^2} = \dfrac{1}{4}$. The limit along this path is $\dfrac{1}{4}$. To show that it is the limit of the function as $(x,y)\to(0,0)$, we use the **Squeeze Theorem**.

$$0 \le \left|f(x,y)-\frac{1}{4}\right| = \left|\frac{x^2-3x^2y+y^2}{4x^2+4y^2} - \frac{1}{4}\right| = \left|\frac{(x^2-3x^2y+y^2)-(x^2+y^2)}{4(x^2+y^2)}\right|$$
$$= \left|\frac{-3x^2y}{4(x^2+y^2)}\right| = \frac{3}{4}\cdot\frac{x^2}{x^2+y^2}|y| \le \frac{3}{4}|y|$$

Since $\lim\limits_{(x,y)\to 0} 0 = 0$, $\lim\limits_{(x,y)\to 0} \dfrac{3}{4}|y| = 0$, we get

$$\lim_{(x,y)\to 0}\left|f(x,y)-\frac{1}{4}\right| = 0 \Longleftrightarrow \lim_{(x,y)\to 0}\left(f(x,y)-\frac{1}{4}\right) = 0 \Longleftrightarrow \lim_{(x,y)\to 0} f(x,y) = \frac{1}{4}$$



10. (8 points) Find the absolute maximum and absolute minimum values of the function

$$f(x, y) = 4x + 4y - x^2 - y^2$$

subject to the constraint $x^2 + y^2 = 2$.

Solution: Let $g(x, y) = x^2 + y^2$. Then the constraint equation is $g(x, y) = 2$.

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(4x + 4y - x^2 - y^2\right) = 4 - 2x & g_x &= \frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x \\ f_y &= \frac{\partial}{\partial y}\left(4x + 4y - x^2 - y^2\right) = 4 - 2y & g_y &= \frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y \\ \nabla f &= \langle f_x, f_y \rangle = \langle 4 - 2x, 4 - 2y \rangle & \nabla g &= \langle g_x, g_y \rangle = \langle 2x, 2y \rangle \end{aligned}$$

By the **Method of Lagrange Multipliers**, we solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y) \quad = 2 \end{cases}$ which is the same as

$$\begin{cases} \langle 4 - 2x, 4 - 2y \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 2 \end{cases} \iff \begin{cases} 4 - 2x = 2\lambda x \; ① \\ 4 - 2y = 2\lambda y \; ② \\ x^2 + y^2 = 2 \quad ③ \end{cases} \iff \begin{cases} (1 + \lambda)x = 2 \; ① \\ (1 + \lambda)y = 2 \; ② \\ x^2 + y^2 = 2 \; ③ \end{cases}$$

From ① and ②, we get $x = y = \dfrac{2}{1 + \lambda}$. By ③, we have $2x^2 = 2$, so $x = \pm 1$, and hence we get only two points $(x, y) = (-1, -1)$ and $(x, y) = (1, 1)$. So we have

$$f(-1, -1) = 4(-1) + 4(-1) - (-1)^2 - (-1)^2 = -10$$
$$f(1, 1) = 4(1) + 4(1) - (1)^2 - (1)^2 = 6$$

So the **absolute max/min values** of the function are $\boxed{\mathbf{6}}$/ $\boxed{\mathbf{-10}}$, respectively.

Solution 2: We may parametrize the constraint curve as $x = \sqrt{2}\cos\theta$, $y = \sqrt{2}\sin\theta$. Then

$$f(x, y) = 4\sqrt{2}\cos\theta + 4\sqrt{2}\sin\theta - 2 \equiv g(\theta), \text{ where } 0 \le \theta < 2\pi$$

To find the extreme values of f, also for g, we solve the critical points of g.

$$g'(\theta) = -4\sqrt{2}\sin\theta + 4\sqrt{2}\cos\theta) = 4\sqrt{2}(-\sin\theta + \cos\theta)$$
$$g'(\theta) = 0 \iff \sin\theta = \cos\theta \iff \tan\theta = 1 \iff \theta = \frac{\pi}{4}, \; \frac{5\pi}{4}$$

As $\theta = \dfrac{\pi}{4}$, $x = \sqrt{2}\cos\dfrac{\pi}{4} = \sqrt{2} \cdot \dfrac{\sqrt{2}}{2} = 1$, $y = \sqrt{2}\sin\dfrac{\pi}{4} = \sqrt{2} \cdot \dfrac{\sqrt{2}}{2} = 1$, We have

$$f(-1, -1) = 4(-1) + 4(-1) - (-1)^2 - (-1)^2 = -10$$

As $\theta = \dfrac{5\pi}{4}$, $x = \sqrt{2}\cos\dfrac{5\pi}{4} = -\sqrt{2} \cdot \dfrac{\sqrt{2}}{2} = -1$, $y = \sqrt{2}\sin\dfrac{5\pi}{4} = -\sqrt{2} \cdot \dfrac{\sqrt{2}}{2} = -1$. We have

$$f(1, 1) = 4(1) + 4(1) - (1)^2 - (1)^2 = 6$$

So the **absolute max/min values** of the function are $\boxed{\mathbf{6}}$/ $\boxed{\mathbf{-10}}$, respectively.

11. (8 points) Let S be a surface given by

$$\vec{r}(u,v) = \left\langle u\cos(v),\ u\sin(v),\ \ln\left(9+u^2\right)\right\rangle$$

where $(u,v) \in [0,2] \times [0,2\pi)$. Find an equation of the tangent plane to S at the point $(0,\ 1,\ \ln(10))$.

Solution: To find the corresponding u and v values of the point $(0,\ 1,\ \ln(10))$, we solve $\vec{r}(u,v) = \langle 0,\ 1,\ \ln(10)\rangle$. That is, To find all the critical points, we solve $\nabla f(x,y) = \langle 0,0\rangle$, or $\langle f_x, f_y\rangle = \langle 0,0\rangle$.

$$\begin{cases} u\cos(v) = 0 & ① \\ u\sin(v) = 1 & ② \\ \ln\left(9+u^2\right) = \ln(10) & ③ \end{cases}$$

From ③, we get $9+u^2 = 10$, so $u^2 = 1$, and we have $u = \pm 1$. Since $u \in [0,2]$, $u = 1$. Substitute it into equations ① and ②, and we get $\cos v = 0$, and $\sin v = 1$. Note $v \in [0,2\pi)$, we get $v = \dfrac{\pi}{2}$. So $(u,v) = \left(1, \dfrac{\pi}{2}\right)$.

$$\vec{r}_u = \frac{\partial}{\partial u}\left\langle u\cos(v),\ u\sin(v),\ \ln\left(9+u^2\right)\right\rangle = \left\langle \cos(v), \sin(v), \frac{2u}{9+u^2}\right\rangle$$

$$\vec{r}_v = \frac{\partial}{\partial v}\left\langle u\cos(v),\ u\sin(v),\ \ln\left(9+u^2\right)\right\rangle = \langle -u\sin(v), u\cos(v), 0\rangle$$

$$\vec{r}_u\left(1, \frac{\pi}{2}\right) = \left\langle \cos\frac{\pi}{2},\ \sin\frac{\pi}{2},\ \frac{2(1)}{9+(1)^2}\right\rangle = \left\langle 0,\ 1,\ \frac{1}{5}\right\rangle$$

$$\vec{r}_v\left(1, \frac{\pi}{2}\right) = \left\langle -(1)\sin\frac{\pi}{2},\ (1)\cos\frac{\pi}{2},\ 0\right\rangle = \langle -1, 0, 0\rangle$$

So the normal direction of the tangent plane is

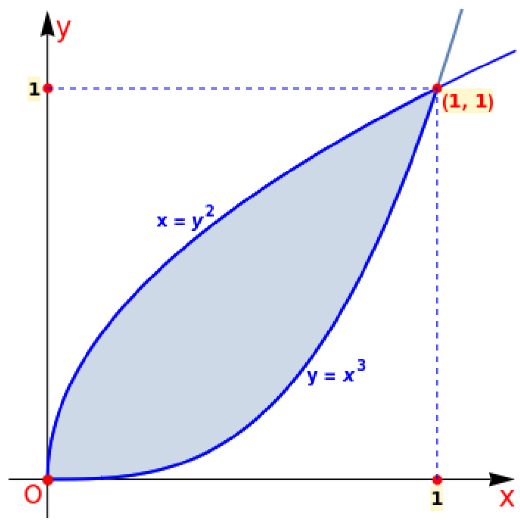
$$\vec{n} = \vec{r}_u\left(1, \frac{\pi}{2}\right) \times \vec{r}_v\left(1, \frac{\pi}{2}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & \frac{1}{5} \\ -1 & 0 & 0 \end{vmatrix} = 0\,\vec{i} - \frac{1}{5}\vec{j} + \vec{k} = \left\langle 0, -\frac{1}{5}, 1\right\rangle$$

So the equation of the tangent plane at $(0,\ 1,\ \ln(10))$ is

$$\boxed{\boldsymbol{(0)(x-0) - \frac{1}{5}(y-1) + (z - \ln(10)) = 0}} \iff \boxed{\boldsymbol{y - 5z - 1 + 5\ln(10) = 0}}$$

12. (8 points) Evaluate the integral $\iint\limits_D 6x^2y^2 \,\mathrm{d}A$, where D is the region bounded by $x = y^2$ and $y = x^3$ in the first quadrant.

Solution: Solving $\begin{cases} x = y^2 \\ y = x^3 \end{cases}$ gives $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$ (point $(-1, -1)$ is not in the first quadrant). In the first quadrant, $x = y^2 \iff y = \sqrt{x}$, $y = x^3 \iff x = x^{\frac{1}{3}}$. So we may sketch the region D as follows,



So $D = \left\{(x, y) \middle| 0 \le y \le 1,\ y^2 \le x \le y^{\frac{1}{3}}\right\}$.

$$\begin{aligned}
\iint\limits_D 6x^2y^2 \,\mathrm{d}A &= \int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} 6x^2y^2 \,\mathrm{d}x\,\mathrm{d}y = \int_0^1 2x^3y^2 \Big|_{x=y^2}^{x=y^{\frac{1}{3}}} \mathrm{d}y \\
&= \int_0^1 2y^2 \left[y - (y^2)^3\right] \mathrm{d}y = \int_0^1 \left(2y^3 - 2y^8\right) \mathrm{d}y \\
&= \left(\frac{y^4}{2} - \frac{2y^9}{9}\right) \Big|_0^1 = \frac{1}{2} - \frac{2}{9} = \boxed{\frac{5}{18}}
\end{aligned}$$

Solution 2: $D = \left\{(x, y) \middle| 0 \le x \le 1,\ y^{\frac{1}{3}} \le x \le \sqrt{x}\right\}$.

$$\begin{aligned}
\iint\limits_D 6x^2y^2 \,\mathrm{d}A &= \int_0^1 \int_{x^3}^{x^{\frac{1}{2}}} 6x^2y^2 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 2x^2y^3 \Big|_{y=x^3}^{y=x^{\frac{1}{2}}} \mathrm{d}x \\
&= \int_0^1 2x^2 \left(x^{\frac{3}{2}} - x^9\right) \mathrm{d}x = \int_0^1 2\left(x^{\frac{7}{2}} - x^{11}\right) \mathrm{d}x \\
&= 2\left(\frac{2}{9}x^{\frac{9}{2}} - \frac{x^{12}}{12}\right) \Big|_0^1 = 2\left(\frac{2}{9} - \frac{1}{12}\right) = \frac{4}{9} - \frac{1}{6} = \boxed{\frac{5}{18}}
\end{aligned}$$

13. (8 points) Evaluate the iterated integral by reversing the order of integration

$$\int_0^1 \int_{x^{1/3}}^1 \frac{1}{y^4+1}\, dy\, dx$$

Solution:

$$\int_0^1 \int_{x^{1/3}}^1 \frac{1}{y^4+1}\, \mathrm{d}y\, \mathrm{d}x = \int_0^1 \int_0^{y^3} \frac{1}{y^4+1}\, \mathrm{d}x\, \mathrm{d}y = \int_0^1 \left. \frac{x}{y^4+1} \right|_{x=0}^{x=y^3} \mathrm{d}y$$

$$= \int_0^1 \frac{y^3}{y^4+1} \mathrm{d}y = \int_1^2 \frac{\mathrm{d}u}{4u} = \frac{1}{4} \ln|u| \Big|_1^2 = \frac{1}{4}(\ln 2 - \ln 1)$$

$$= \boxed{\frac{\ln 2}{4}}$$