Math 2400, Midterm 3

April 16, 2018

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50
☐	Section 002	Xingzhou Yang	8:00–8:50
☐	Section 003	Albert Bronstein	9:00–9:50
☐	Section 004	Cliff Blakestad	10:00–10:50
☐	Section 005	Albert Bronstein	10:00–10:50
☐	Section 006	Mark Pullins	11:00–11:50
☐	Section 009	Taylor Klotz	11:00–11:50
☐	Section 007	Albert Bronstein	12:00–12:50
☐	Section 008	Martin Walter	1:00–1:50
☐	Section 010	Braden Balentine	2:00–2:50
☐	Section 011	Pedro Berrizbeitia	3:00–3:50
☐	Section 012	Pedro Berrizbeitia	4:00–4:50

Question	Points	Score
1	12	
2	12	
3	8	
4	12	
5	16	
6	12	
7	12	
8	16	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like **100/7** or expressions like **ln(3)/2** as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (12 points) Match the vector fields $\vec{F}$ with the plots below.

(1) $\vec{F} = \langle y, x \rangle$ (C)

(2) $\vec{F} = \langle x, y \rangle$ (B)

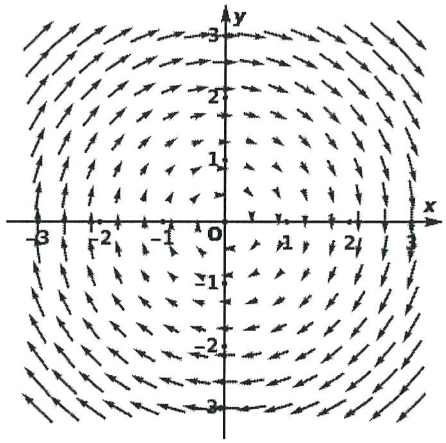
(3) $\vec{F} = \langle y, -x \rangle$ (A)

(4) $\vec{F} = \langle x, -y \rangle$ (F)

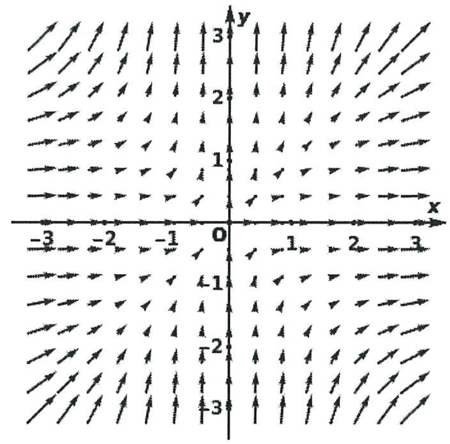
(5) $\vec{F} = \langle x^2, y^2 \rangle$ (D)

(6) $\vec{F} = \langle 2 - x, 1 - y \rangle$ (E)

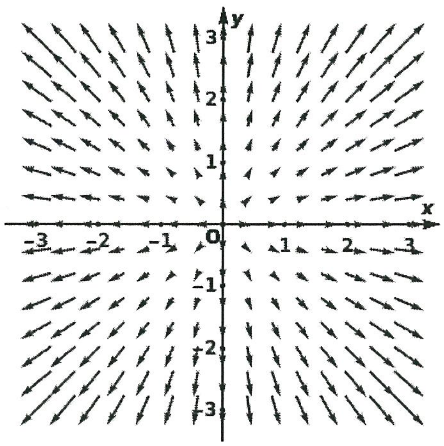
(A)



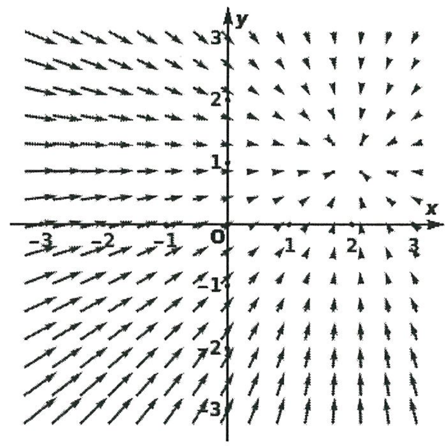
(D)



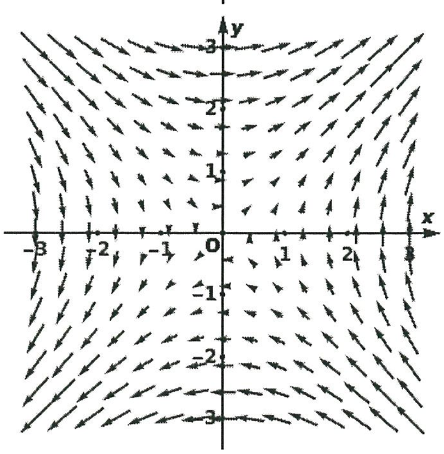
(B)



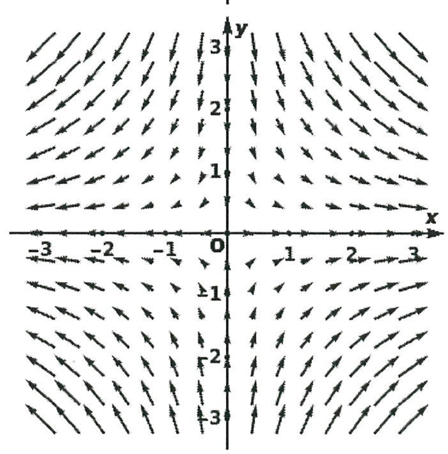
(E)



(C)

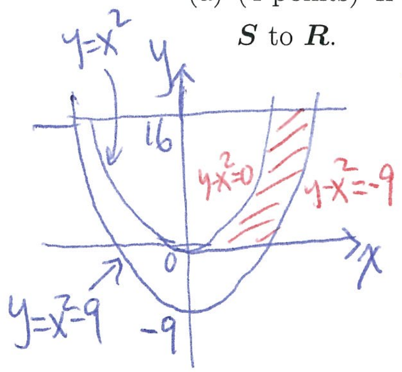


(F)



2. Consider the transformation $T^{-1} : R \to S$ given by $T^{-1}(x, y) = (y, y - x^2)$, where R is the region in the first quadrant of the xy-plane bounded by $y = 0$, $y = 16$, $y = x^2$, and $y = x^2 - 9$.

(a) (4 points) If $u = y$ and $v = y - x^2$, find the transformation $T(u, v)$ from S to R.



$\begin{cases} u = y & (1) \\ v = y - x^2 & (2) \end{cases}$ plug (1) into (2) $\Rightarrow v = u - x^2$

$x^2 = u - v$, in the 1rst quadrant

$x \geq 0$, $x = \sqrt{u-v}$

So we get $T : S \to R$

$\boxed{\begin{cases} x = \sqrt{u-v} \\ y = u \end{cases}}$ $\boxed{T(u, v) = (\sqrt{u-v}, u)}$

(b) (4 points) Calculate the Jacobian, $\dfrac{\partial(x, y)}{\partial(u, v)}$, of the transformation T.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{u-v}} & -\frac{1}{2\sqrt{u-v}} \\ 1 & 0 \end{vmatrix} = \boxed{\frac{1}{2\sqrt{u-v}}}$$

(c) (4 points) Use transformation T to evaluate $\displaystyle\iint_R x \, dA$.

$$\iint_R x \, dA = \iint_S \sqrt{u-v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| d\tilde{A}$$

$$= \int_{-9}^{0} \int_{0}^{16} \sqrt{u-v} \cdot \frac{1}{2\sqrt{u-v}} \, dv \, du$$

$$= \int_{-9}^{0} \int_{0}^{16} \frac{1}{2} \, dv \, du = \frac{1}{2}(16-0)(0-(-9))$$

$$= \boxed{72}$$

3. (8 points) Select the integral that is ALWAYS equivalent to the integral given. No work is required.

(1) $\displaystyle\int_0^1 \int_{y^2+2}^{\sqrt{y}+2} \int_0^{x+y} f(x,y,z)\,dz\,dx\,dy$

$E=\{(x,y,z) \mid 0\le z\le x+y,\ (x,y)\in D_{xy}\}$

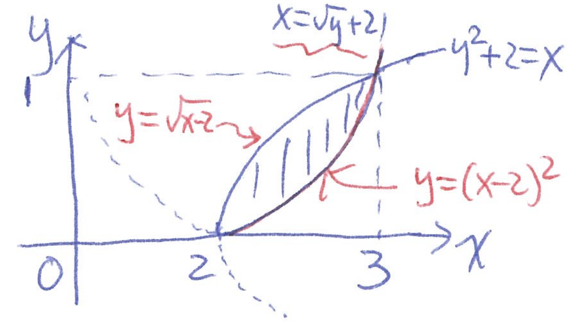
$D_{xy}=\{(x,y) \mid 0\le y\le 1,\ y^2+2\le x\le \sqrt{y}+2\}$

(A) $\displaystyle\int_0^1 \int_{y^2+2}^{\sqrt{y}+2} \int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$

(B) $\displaystyle\int_0^1 \int_{\sqrt{x-2}}^{(x-2)^2} \int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$

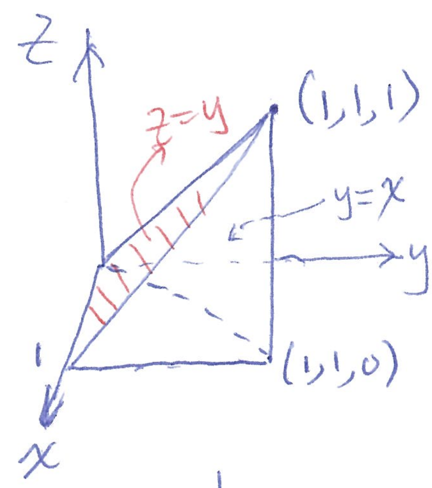
(C) $\displaystyle\int_2^3 \int_{\sqrt{x-2}}^{(x-2)^2} \int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$

(D) $\displaystyle\int_2^3 \int_{(x-2)^2}^{\sqrt{x-2}} \int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$



$D_{xy}=\{(x,y) \mid 2\le x\le 3,\ (x-2)^2\le y\le \sqrt{x-2}\}$

(2) $\displaystyle\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx$



(A) $\displaystyle\int_0^1 \int_1^z \int_1^x f(x,y,z)\,dy\,dx\,dz$

(B) $\displaystyle\int_0^1 \int_z^1 \int_x^1 f(x,y,z)\,dy\,dx\,dz$

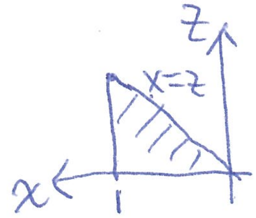
(C) $\displaystyle\int_0^1 \int_0^1 \int_x^z f(x,y,z)\,dy\,dx\,dz$

(D) $\displaystyle\int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz$

$E=\{(x,y,z) \mid z\le y\le x,\ (x,z)\in D_{xz}\}$

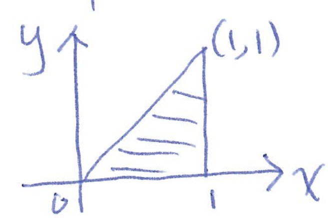
$D_{xz}=$ Projection of E onto xz-plane

$=\{(x,z) \mid 0\le z\le 1,\ z\le x\le 1\}$

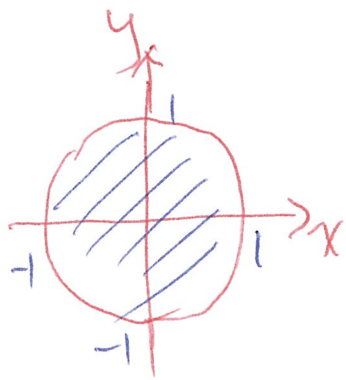


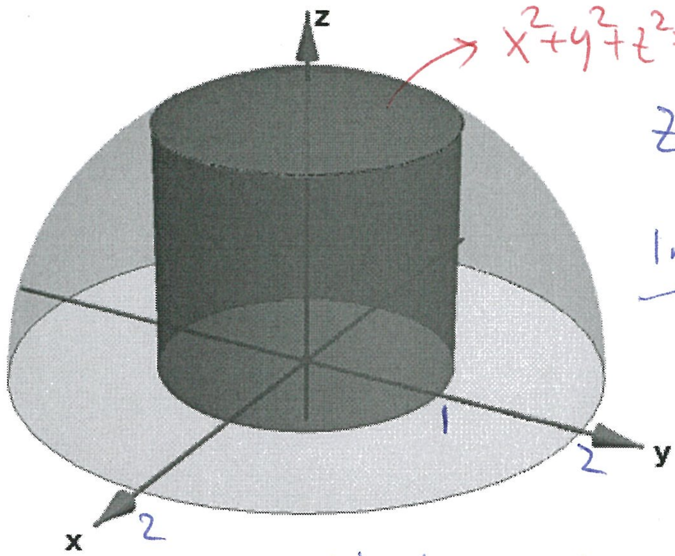
$E=\{(x,y,z) \mid 0\le z\le y,\ (x,y)\in D_{xy}\}$

$D_{xy}=\{(x,y) \mid 0\le x\le 1,\ 0\le y\le x\}$



4. The solid $\boldsymbol{E}$ is inside the cylinder $\boldsymbol{x^2+y^2=1}$, below the sphere $\boldsymbol{x^2+y^2+z^2=4}$, and above the plane $\boldsymbol{z=0}$. The density of the solid is $\boldsymbol{\rho(x,y,z)=\sqrt{x^2+y^2}}$.





$x^2+y^2+z^2=4$

$z=\sqrt{4-x^2-y^2}$

In Cylindrical Coordinate

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$

$x^2+y^2=r^2$

vol. element $dV=r\cdot dz\,dr\,d\theta$

(a) (6 points) Set up, but do **not** evaluate, the integral for the mass of the solid.

The mass $m=\iiint_E \rho(x,y,z)\,dV=\iint_{D_{xy}}\int_0^{\sqrt{4-x^2-y^2}}\rho(x,y,z)\,dz\,dA$

$=\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\cdot \underbrace{r\,dz\,dr\,d\theta}_{dv}$

(b) (6 points) If $(\bar{x},\,\bar{y},\,\bar{z})$ is the center of mass of the solid, set up, but do **not** evaluate, the integral for $\bar{z}$.

$\bar{z}=\frac{1}{m}\iiint_E x\,\rho(x,y,z)\,dV$

$=\frac{1}{m}\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r\cos\theta\cdot r\cdot r\,dz\,dr\,d\theta$

The Projection of E onto xy plane is a disk with radius $R=1$

5. Let S be the parametric surface $\vec{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$ for $0 \le u \le \sqrt{2}$ and $0 \le v \le u$.

(a) (8 points) Compute $\vec{r}_u \times \vec{r}_v$.

$\vec{r}_u = \langle \cos(v), \sin(v), 0\rangle, \quad \vec{r}_v = \langle -u\sin(v), u\cos(v), 1\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} \sin(v) & 0 \\ u\cos(v) & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} \cos(v) & 0 \\ -u\sin(v) & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} \cos(v) & \sin(v) \\ -u\sin(v) & u\cos(v) \end{vmatrix}$$

$$= \sin(v)\vec{i} - \cos(v)\vec{j} + [u\cos^2(v) + u\sin^2(v)]\vec{k}$$

$$= \boxed{\langle \sin(u), -\cos(v), u\rangle}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2(u) + (-\cos(v))^2 + u^2} = \sqrt{1+u^2}$$

(b) (8 points) Determine the surface area of S.

$$A(S) = \iint_{D_{uv}} |\vec{r}_u \times \vec{r}_v|\, dA = \iint_D \sqrt{(\sin(u))^2 + (-\cos(u))^2 + u^2}\, dA$$

$$= \int_0^{\sqrt{2}}\int_0^u \sqrt{1+u^2}\, dv\, du$$

$$= \int_0^{\sqrt{2}} \sqrt{1+u^2}\, v\Big|_{u=0}^{u=u} du = \int_0^{\sqrt{2}} u\sqrt{1+u^2}\, du$$

$$= \int_1^{\sqrt{3}} t^2\, dt$$

$$= \frac{t^3}{3}\Big|_1^{\sqrt{3}} = \frac{(\sqrt{3})^3}{3} - \frac{1}{3}$$

$$= \boxed{\sqrt{3} - \frac{1}{3}}$$

$t = \sqrt{1+u^2}$

$t^2 = 1+u^2$

$d(t^2) = d(1+u^2)$

$2t\,dt = 2u\,du$

$t\,dt = u\,du$

$u\sqrt{1+u^2}\,du = t(t\,dt) = t^2\,dt$

u	0	$\sqrt{2}$
t	1	$\sqrt{3}$

6. (12 points) Evaluate $\int_C (x^2+y^2+z^2)\,ds$ where

$$C:\ x=\sqrt{8}\,t,\ y=\cos t,\ z=\sin t,\ 0\le t\le 1.$$

let $\vec{r}(t)=\langle x,y,z\rangle=\langle \sqrt{8}t, \cos t, \sin t\rangle$

$\vec{r}\,'(t)=\langle \sqrt{8}, -\sin t, \cos t\rangle$

$|\vec{r}\,'(t)|=\sqrt{8+(-\sin t)^2+\cos^2 t}=\sqrt{8+\sin^2 t+\cos^2 t}=\sqrt{9}=3$

$ds=|\vec{r}\,'(t)|dt=3dt$

$x^2+y^2+z^2=(\sqrt{8}t)^2+(\cos t)^2+(\sin t)^2$

$=8t^2+\cos^2 t+\sin^2 t=8t^2+1$

$\int_C (x^2+y^2+z^2)ds=\int_0^1 (8t^2+1)3dt$

$=3\left(\frac{8}{3}t^3+t\right)\Big|_0^1=3\left(\frac{8}{3}+1\right)=3\left(\frac{11}{3}\right)$

$=\boxed{11}$

7. (12 points) Let $F = \langle y, x + e^z, ye^z + 1\rangle$. Calculate the integral $\int_C F \cdot dr$ where C is given by $r(t) = \langle t^2, t^3, \sin(\pi t)\rangle$ for $0 \le t \le 2$.

Soln 1: $\vec{r}\,'(t) = \langle 2t, 3t^2, \pi\cos(\pi t)\rangle$

$\vec{F}(\vec{r}(t)) = \langle t^3, t^2 + e^{\sin(\pi t)}, t^3 e^{\sin(\pi t)} + 1\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = (2t)(t^3) + 3t^2(t^2 + e^{\sin(\pi t)}) + \pi\cos(\pi t)[t^3 e^{\sin(\pi t)} + 1]$

$= 5t^4 + \pi\cos(\pi t) + 3t^2 e^{\sin(\pi t)} + \pi t^3 e^{\sin(\pi t)}\cos(\pi t)$

let $g(t) = t^3 e^{\sin(\pi t)}$, then $g'(t) = 3t^2 e^{\sin(\pi t)} + \pi t^3 e^{\sin(\pi t)}\cos(\pi t)$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 5t^4 + \pi\cos(\pi t) + g'(t)$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^2 [5t^4 + \pi\cos(\pi t) + g'(t)]\,dt$

$= [t^5 + \sin(\pi t) + g(t)]\Big|_0^2 = [2^5 + \sin(2\pi) + 2^3 e^{\sin(2\pi)}] - [0+0+0]$

$= 32 + 0 + 8 = \boxed{40}$

Soln 2: let $\vec{F} = \nabla f$. Solve it for f (to see if $\vec{F}$ is conservative)

So we get $\begin{cases} f_x = y & ① \\ f_y = x + e^z & ② \\ f_z = ye^z + 1 & ③ \end{cases}$ ① $\Rightarrow f(x,y,z) = \int f_x\,dx = \int y\,dx = xy + g(y,z)$ — ④

plug it into ②

$\Rightarrow \frac{\partial}{\partial y}(xy + g(y,z)) = x + e^z \Leftrightarrow x + g_y = x + e^z \Rightarrow g_y = e^z$

$g(y,z) = \int g_y(y,z)\,dy = \int e^z\,dy = ye^z + h(z)$ ⑤

plug ⑤ into ④ $\Rightarrow f(x,y,z) = xy + ye^z + h(z)$ ⑥

plug ⑥ into ③ $\Rightarrow f_z = \frac{\partial}{\partial z}(xy + ye^z + h(z)) = ye^z + 1$

$\Rightarrow ye^z + h'(z) = ye^z + 1$ so $h'(z) = 1$ so $h(z) = z + C$

So $f(x,y,z) = xy + ye^z + z + C$ → next page.

Cont'd (soln to problem 7)

Check if f is a potential of $\vec{F}$

$f_x = \frac{\partial}{\partial x}(xy + ye^z + z + c) = y$, $f_y = \frac{\partial}{\partial y}(xy + ye^z + z + c) = x + e^z$

$f_z = \frac{\partial}{\partial z}(xy + ye^z + z + c) = ye^z + 1$

So $\nabla f = \langle f_x, f_y, f_z \rangle = \langle y, x+e^z, ye^z+1 \rangle = \vec{F}$.

as $t=0$ $\vec{r}(0) = \langle 0, 0, \sin(0) \rangle = \langle 0, 0, 0 \rangle$

as $t=2$ $\vec{r}(2) = \langle 2^2, 2^3, \sin(2\pi) \rangle = \langle 4, 8, 0 \rangle$

By FTLIs, $\int_C \vec{F} \cdot d\vec{r} = f(4,8,0) - f(0,0,0)$

$= [(4)(8) + (8)e^0 + 0] - [(0)(0) + 0 \cdot e^0 + 0]$

$= 32 + 8 = \boxed{40}$

Solution 3:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x+e^z & ye^z+1 \end{vmatrix} = \vec{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+e^z & ye^z+1 \end{vmatrix} - \vec{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y & ye^z+1 \end{vmatrix} + \vec{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y & x+e^z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(ye^z+1) - \frac{\partial}{\partial z}(x+e^z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(ye^z+1) - \frac{\partial}{\partial z}(y)\right) + \vec{k}\left(\frac{\partial}{\partial x}(x+e^z) - \frac{\partial}{\partial y}(y)\right)$$

$$= \vec{i}(e^z - e^z) - \vec{j}(0-0) + \vec{k}(1-1) = \langle 0, 0, 0 \rangle$$

So $\vec{F}$ is conservative. So $\int_C \vec{F} \cdot d\vec{r}$ is indep. of path.

Choose a line segment from (0,0,0) to (4,8,0) as a path.

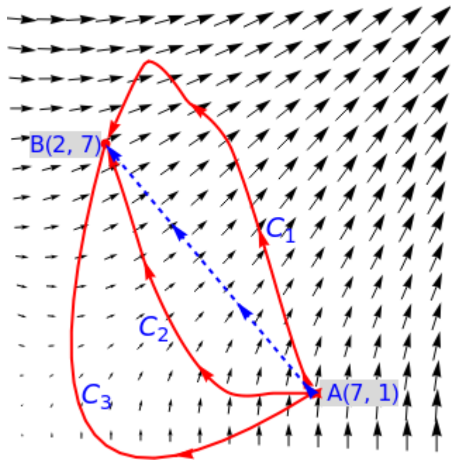
$\vec{r}(t) = (1-t)\langle 0,0,0 \rangle + t\langle 4,8,0 \rangle = \langle 4t, 8t, 0 \rangle$, $0 \le t \le 1$

$\vec{F}(\vec{r}(t)) = \langle 8t, 4t+1, 8t+1 \rangle$, $\vec{r}\,'(t) = \langle 4, 8, 0 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 (64t + 8)\, dt$$

$$= (32t^2 + 8t)\Big|_0^1 = 32 + 8 = \boxed{40}$$

8. The figure below shows the vector field $\vec{F}(x, y) = \langle y, x + \cos y \rangle$, and 3 curves C_1, C_2, C_3 start at $A(7, 1)$ and end at $B(2, 7)$.



(a) (4 points) Determine if $\vec{F}$ is conservative. Justify.

Solution: Let $P(x, y) = y$, $Q(x, y) = x + cos(y)$. Then

$$Q_x = \frac{\partial}{\partial x}(x + \cos(y)) = 1, \quad P_y = \frac{\partial}{\partial y}(y) = 1$$

So $Q_x = P_y$ and $\vec{F}$ is conservative.

(b) (4 points) Explain why $\int_C \vec{F} \cdot d\vec{r}$ has the same value for all three paths.

Solution: From (a), $\vec{F}$ is conservative, and so $\vec{F} \cdot d\vec{r}$ is independent of path. C_1, C_2, and C_3 have the same initial and terminal points. So

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r} = f(B) - f(A),$$

where f is a potential function of $\vec{F}$.

(c) (8 points) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is a smooth path from A to B.

Solution 1: Let f be a potential function of $\vec{F}$. Then we have $f_x = y$, $f_y = x + \cos(y)$. From the first equation we have

$$f(x, y) = \int f_x \, dx = \int y \, dx = xy + g(y) \qquad ①$$

Plugging ① to $f_y = x + \cos(y)$ gives

$$f_y = \frac{\partial}{\partial y}(xy + g(y)) = x + g'(y) = x + \cos(y) \qquad ②$$

So we get $g'(y) = \cos(y)$, and so $g(y) = \int \cos(y) dy = \sin(y) + C$. From ①, $f(x, y) = xy + \sin(y) + C$. So

$$\int_C \vec{F} \cdot d\vec{r} = f(2, 7) - f(7, 1) = [(2)(7) + \sin(7)] - [(7)(1) + \sin(1)]$$
$$= 7 - \sin 1 + \sin(7).$$

Solution 2: From (a), $\vec{F}$ is conservative, $\vec{F} \cdot d\vec{r}$ is independent of path. We choose the line segment from A to B as a path C, whose equation is $\vec{r}(t) = (1-t)\langle 7, 1\rangle + t\langle 2, 7\rangle = \langle 7-5t, 6t+1\rangle$. $0 \le t \le 1$. $\vec{r}\,'(t) = \langle -5, 6\rangle$.

$$\begin{aligned}
\vec{F}(\vec{r}(t)) &= \langle y, x+\cos y\rangle = \langle 6t+1, -5t+\cos(6t+1)+7\rangle \\
\vec{F} \cdot d\vec{r} &= \vec{F} \cdot \vec{r}\,'(t)\, dt \\
&= [(-5)(6t+1) + (6)(-5t+\cos(6t+1)+7)]\, dt \\
&= [-60t + 6\cos(6t+1) + 37]\, dt
\end{aligned}$$

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 [-60t + 6\cos(6t+1) + 37]\, dt \\
&= \left[-30t^2 + 37t + \sin(6t+1)\right]\Big|_0^1 \\
&= 7 - \sin 1 + \sin(7).
\end{aligned}$$