Math 2400, Midterm 1

February 12, 2018

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50
☐	Section 002	Xingzhou Yang	8:00–8:50
☐	Section 003	Albert Bronstein	9:00–9:50
☐	Section 004	Cliff Blakestad	10:00–10:50
☐	Section 005	Albert Bronstein	10:00–10:50
☐	Section 006	Mark Pullins	11:00–11:50
☐	Section 009	Taylor Klotz	11:00–11:50
☐	Section 007	Albert Bronstein	12:00–12:50
☐	Section 008	Martin Walter	1:00–1:50
☐	Section 010	Braden Balentine	2:00–2:50
☐	Section 011	Pedro Berrizbeitia	3:00–3:50
☐	Section 012	Pedro Berrizbeitia	4:00–4:50

Question	Points	Score
1	14	
2	14	
3	15	
4	15	
5	15	
6	15	
7	12	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like $\mathbf{100/7}$ or expressions like $\mathbf{\ln(3)/2}$ as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (14 points) Find an equation of the plane that contains the points $(-2,3,1)$, $(1,0,2)$, and $(1,2,-1)$.

Solution: Denote the points by $A(-2,3,1)$, $B(1,0,2)$, $C(1,2,-1)$.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 1,0,2\rangle - \langle -2,3,1\rangle = \langle 3,-3,1\rangle$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \langle 1,2,-1\rangle - \langle -2,3,1\rangle = \langle 3,-1,-2\rangle$$
$$\begin{aligned}\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -3 & 1 \\ 3 & -1 & -2 \end{vmatrix} \\ &= \begin{vmatrix} -3 & 1 \\ -1 & -2 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 1 \\ 3 & -2 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & -3 \\ 3 & -1 \end{vmatrix}\vec{k} \\ &= (6+1)\vec{i} - (-6-3)\vec{j} + (-3+9)\vec{k} = \langle 7,9,6\rangle\end{aligned}$$

$\vec{n} = \langle 7,9,6\rangle$ is normal to the plane that contains $A(-2,3,1)$, $B(1,0,2)$, $C(1,2,-1)$. So the equation of the plane is

$$\begin{aligned}&\langle 7,9,6\rangle \cdot (\langle x,y,z\rangle - \langle -2,3,1\rangle) = 0 \\ &\text{or } 7(x+2) + 9(y-3) + 6(z-1) = 0 \\ &\text{or } 7x + 9y - 6z - 19 = 0;\end{aligned}$$

2. (14 points) Find the volume of the parallelipiped determined by the following three vectors: $\vec{a} = \langle 3, 3, 3 \rangle$, $\vec{b} = \langle 2, 3, 0 \rangle$, and $\vec{c} = \langle 0, 1, 1 \rangle$.

Solution: The volume of the parallelipiped determined by $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\text{Volume} = \left| \overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) \right| \overset{\text{or}}{=} \left| (\overrightarrow{a} \times b) \cdot \overrightarrow{c}) \right| \overset{\text{or}}{=} \left| (\overrightarrow{c} \times a) \cdot \overrightarrow{b}) \right|$$

$$\overrightarrow{b} \times \overrightarrow{c} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} \overrightarrow{i} - \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \overrightarrow{j} + \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} \overrightarrow{k}$$

$$= (3-0)\overrightarrow{i} - (2-0)\overrightarrow{j} + (2-0)\overrightarrow{k} = \langle 3, -2, 2 \rangle$$

$$\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = \langle 3, 3, 3 \rangle \cdot \langle 3, -2, 2 \rangle = 3 \cdot 3 + 3 \cdot (-2) + 3 \cdot 2 = 9$$

$$\text{Volume} = \left| \overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) \right| = 9$$

Note 1: The scalar triple product can also be computed by the determinant directly.

$$\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = \begin{vmatrix} 3 & 3 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 3(3-0) - 3(2-0) + 3(2-0) = 9$$

3. Consider the curve in space described by $\vec{r}(t) = \langle t, 2\ \sin(t), 2\ \cos(t)\rangle$.

(a) (8 points) Find the equation of the line tangent to this curve when $t = \pi$.

Solution:

$$\begin{aligned}
\vec{r}\,'(t) &= \frac{d}{dt}\langle t,\ 2\sin(t),\ 2\cos(t)\rangle = \langle 1,\ 2\cos(t),\ -2\sin(t)\rangle \\
\vec{r}\,'(\pi) &= \langle 1,\ 2\cos(\pi),\ -2\sin(\pi)\rangle = \langle 1,\ -2,\ 0\rangle \\
\vec{r}\,(\pi) &= \langle \pi,\ 2\sin(\pi),\ \cos(\pi)\rangle = \langle \pi,\ 0,\ -2\rangle
\end{aligned}$$

So the equation of the tangent line is

$$\begin{aligned}
\vec{r} &= \vec{r}\,(\pi) + t\,\vec{r}\,'(\pi) \\
\text{or } \vec{r} &= \langle \pi,\ 0,\ -2\rangle + t\ \langle 1,\ -2,\ 0\rangle = \langle \pi + t,\ -2t,\ -2\rangle
\end{aligned}$$

where $t \in \mathbb{R}$ is the parameter.

You may also write the equation of the tangent line in parametric equations or in symmetric form as follows,

$$\begin{cases} x &= \pi + t \\ y &= -2t \\ z &= -2 \end{cases} \quad t \in \mathbb{R} \text{ is the parameter.} \qquad \text{or} \qquad \frac{x-\pi}{1} = \frac{y-0}{-2} = \frac{z+2}{0}$$

(b) (7 points) Find the length of the arc of this curve between the points $(0, 0, 2)$ and $(\pi, 0, -2)$

Solution: It is easy to see that point $(0, 0, 2)$ corresponds to $t = 0$, point $(\pi, 0, -2)$ corresponds to $t = \pi$.

$$\begin{aligned}
L &= \int_0^\pi \left|\vec{r}\,'(t)\right|\ dt = \int_0^\pi \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \\
&= \int_0^\pi \sqrt{(1)^2 + [2\sin(t)]^2 + [-2\cos(t)]^2}\, dt \\
&= \int_0^\pi \sqrt{1 + 4[\sin^2(t) + \cos^2(t)]}\, dt \\
&= \int_0^\pi \sqrt{1+4}\, dt = \int_0^\pi \sqrt{5}\, dt = \sqrt{5}t\Big|_0^\pi = \sqrt{5}\pi.
\end{aligned}$$

4. Assume that the lines

$$L_1: \qquad x = 1 + t, \;\; y = 1 + 6t, \;\; z = 2t$$

$$L_2: \qquad x = 1 + 2s, \;\; y = 5 + 15s, \;\; z = -2 + 6s$$

are skew.

(a) (5 points) Find a vector $\vec{n}$ normal to both L_1 and L_2.

Solution: The direction vectors for L_1 and L_2 are, respectively,

$$\vec{n_1} = \langle 1,\, 6,\, 2\rangle,\ \vec{n_2} = \langle 2,\, 16,\, 6\rangle.$$

$\vec{n} = \vec{n_1} \times \vec{n_2}$ is normal to $\vec{n_1}$ and $\vec{n_2}$, and so is to L_1 and L_2.

$$\vec{n} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & 2 \\ 2 & 16 & 6 \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 16 & 6 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 6 \\ 2 & 16 \end{vmatrix} \vec{k}$$

$$= (36 - 30)\,\vec{i} - (6 - 4)\,\vec{j} + (16 - 12)\,\vec{k} = \langle 6,\, -2,\, 3\rangle$$

(b) (5 points) Find an equation of the plane that contains L_2 and normal to $\vec{n}$.

Solution: Let $s = 0$ and we get a point $Q_0\,\langle 1,\, 5,\, -2\rangle \in L_2$. From the result in (a), we have the equation of the plane is

$$6(x-1) + (-2)(y-5) + 3(z-(-2)) = 0$$

$$\text{or } 6x - 2y + 3z + 10 = 0$$

(c) (5 points) Find the distance between L_1 and L_2.

Solution: We denote the plane in (b) by π. Pick any point on L_1. For example, let $t = 0$, and we get $P_0\,\langle 1,\, 1,\, 0\rangle \in L_1$. Then

$$\text{dist}(L_1, L_2) = \text{dist}(Q_0, \pi) = \frac{|6(1) + (-2)(1) + 3(0) + 10|}{\sqrt{(6)^2 + (-2)^2 + 3)^2}} = \frac{14}{\sqrt{49}} = 2$$

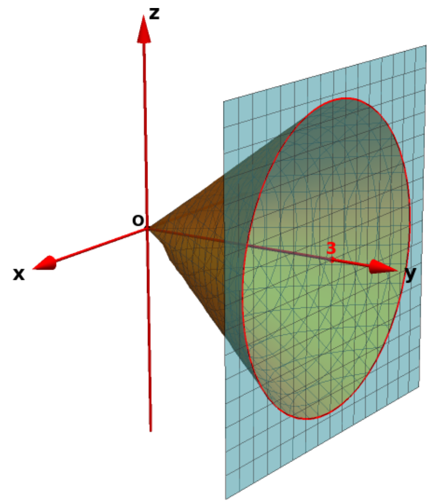
Solution 2: Note $P_0\,(1,\, 1,\, 0) \in L_1$, $Q_0\,(1,\, 5,\, -2) \in L_2$, and $\overrightarrow{P_0Q_0} = \langle 0,\, 4,\, -2\rangle$.

$$\text{dist}(L_1, L_2) = \left|\text{Proj}_{\vec{n}}\overrightarrow{P_0Q_0}\right| = \left|\frac{\overrightarrow{P_0Q_0}\cdot\vec{n}}{|\vec{n}|}\cdot\frac{\vec{n}}{|\vec{n}|}\right| = \left|\frac{\overrightarrow{P_0Q_0}\cdot\vec{n}}{|\vec{n}|}\right|$$

$$= \left|\text{Comp}_{\vec{n}}\overrightarrow{P_0Q_0}\right| = \frac{|\langle 0,\, 4,\, -2\rangle\cdot\langle 6,\, -2,\, 3\rangle|}{\sqrt{(6)^2 + (-2)^2 + 3)^2}} = \frac{14}{\sqrt{49}} = 2$$

5. (15 points) Find a parametric representation for the cone $\boldsymbol{y^2 = 2x^2 + 2z^2}$ between the planes $\boldsymbol{y = 0}$ and $\boldsymbol{y = 3}$.

Solution: Solve the cone equation for $\boldsymbol{y}$ and we have $\boldsymbol{y = \pm\sqrt{2x^2 + 2z^2}}$. Since $\boldsymbol{0 \le y \le 3}$, $\boldsymbol{y = \sqrt{2x^2 + 2z^2}}$. There are at least 3 different ways to parametrize the equation.



Method 1: Let $\boldsymbol{x = x}$, and $\boldsymbol{z = z}$. Then $\boldsymbol{y = \sqrt{2x^2 + 2z^2}}$. Since $\boldsymbol{0 \le y \le 3}$, $\boldsymbol{0 \le \sqrt{2x^2 + 2z^2} \le 3}$, i.e., $\boldsymbol{2x^2 + 2z^2 \le 9}$, or $\boldsymbol{x^2 + z^2 \le \frac{9}{2}}$.

$$\left\{\begin{array}{lcl} x & = & x \\ y & = & \sqrt{2x^2 + 2z^2} \\ z & = & z \end{array}\right. \qquad \begin{array}{c} x, z \text{ are parameters} \\ x^2 + z^2 \le \dfrac{9}{2} \end{array}$$

or in vector form, $\boldsymbol{\vec{r}(x, z) = \left\langle x,\ \sqrt{2x^2 + 2z^2},\ z \right\rangle}$.

Method 2: Use cylindrical coordinates, $\boldsymbol{x = r\cos\theta}$, $\boldsymbol{z = r\sin\theta}$, $\boldsymbol{y = y}$. Plug them into the cone equation and we have

$$y = \sqrt{2x^2 + 2z^2} = \sqrt{2(r\cos\theta)^2 + 2(r\sin\theta)^2} = \sqrt{2}r.$$

Since $\boldsymbol{0 \le y \le 3}$, $\boldsymbol{0 \le \sqrt{2}r \le 3}$, and so $\boldsymbol{0 \le r \le \frac{3}{\sqrt{2}}}$.

$$\left\{\begin{array}{lcl} x & = & r\cos\theta \\ y & = & \sqrt{2}\,r \\ z & = & r\sin\theta \end{array}\right. \qquad \begin{array}{c} r, \theta \text{ are parameters} \\ 0 \le r \le \dfrac{3}{\sqrt{2}},\ 0 \le \theta \le 2\pi \end{array}$$

or in vector form, $\boldsymbol{\vec{r}(r, \theta) = \left\langle r\cos\theta,\ \sqrt{2}\,r,\ r\sin\theta \right\rangle}$.

Method 3: Use spherical coordinates, $x = \rho\sin\phi\cos\theta$, $z = \rho\sin\phi\sin\theta$, $y = \rho\cos\phi$. Plug them into the cone equation and we have

$$\begin{aligned} \rho^2\cos^2\phi &= 2\left(\rho\sin\phi\cos\theta\right)^2 + 2\left(\rho\sin\phi\sin\theta\right)^2 \\ &= 2\rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right) = 2\rho^2\sin^2\phi \end{aligned}$$

Dividing both sides by $2\rho^2\cos^2\phi$ gives

$\tan^2\phi = \frac{1}{2}$, or $\tan\phi = \pm\frac{1}{\sqrt{2}}$. Since $0 \le y \le 3$, the cone equation becomes $\tan\phi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. By trig. identities, $\sin\phi = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, $\cos\phi = 3\sqrt{\frac{3}{2}} = \frac{3\sqrt{6}}{2}$. So we have

$$\begin{cases} x = \dfrac{\sqrt{3}}{3}\rho\cos\theta \\ y = \dfrac{\sqrt{6}}{\rho} \\ z = \dfrac{\sqrt{3}}{3}\rho\sin\theta \end{cases} \qquad \begin{array}{c} \rho, \theta \text{ are parameters} \\ 0 \le \rho \le \dfrac{3\sqrt{6}}{2},\ 0 \le \theta \le 2\pi \end{array}$$

or in vector form, $\overrightarrow{r}(\rho,\theta) = \left\langle \frac{\sqrt{3}}{3}\rho\cos\theta,\ \frac{\sqrt{6}}{\rho},\ \frac{\sqrt{3}}{3}\rho\sin\theta \right\rangle$.

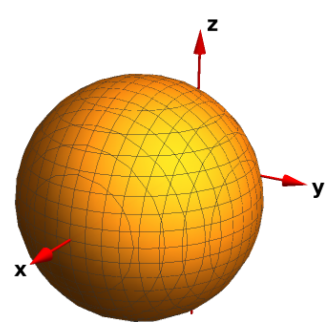
6. (15 points) Find a rectangular (Cartesian) equation for the surface whose spherical equation is

$$\rho = 2\sin\phi\cos\theta$$

Solution: In spherical coordinates, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, $x^2 + y^2 + z^2 = \rho^2$.

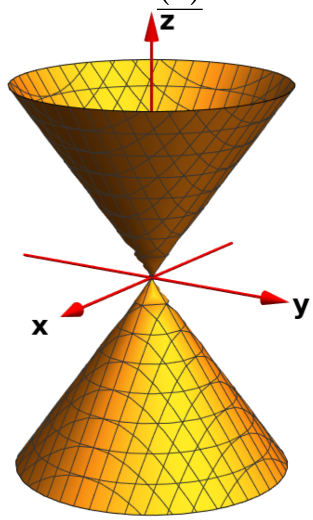
Multiplying the given equation by ρ gives

$$\begin{aligned} &\rho\cdot\rho = 2\rho\sin\phi\cos\theta \\ &x^2 + y^2 + z^2 = 2x \\ &(x-1)^2 + y^2 + z^2 = 1 \end{aligned}$$

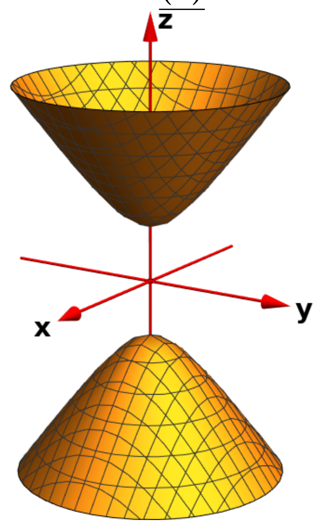


7. (12 points) Match each 3D surface with one of the equations on the right side. Not all equations will be matched.

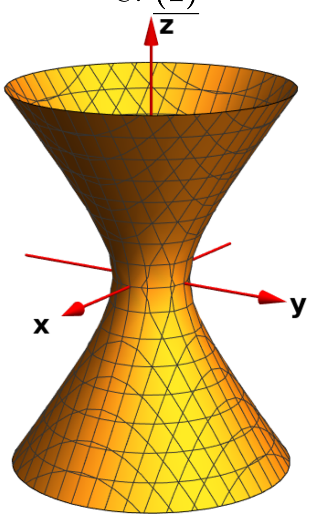
A. **(5)**



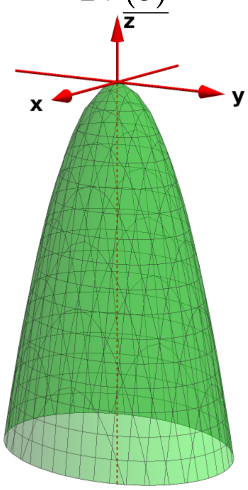
B. **(3)**



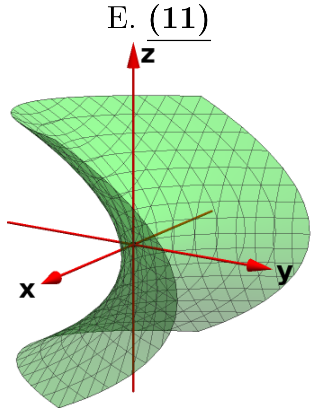
C. **(1)**



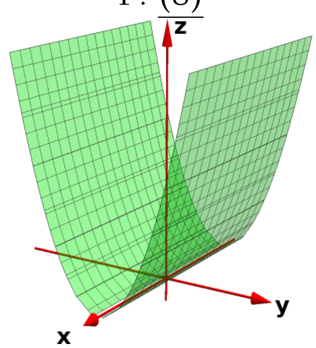
D. **(9)**



E. **(11)**



F. **(8)**



(1) $\boldsymbol{x^2 + y^2 - z^2 = 1}$

(2) $\boldsymbol{x^2 - y^2 + z^2 = 1}$

(3) $\boldsymbol{x^2 + y^2 - z^2 = -1}$

(4) $\boldsymbol{x^2 - y^2 + z^2 = -1}$

(5) $\boldsymbol{x^2 + y^2 - z^2 = 0}$

(6) $\boldsymbol{x^2 - y^2 + z^2 = 0}$

(7) $\boldsymbol{z = x^2}$

(8) $\boldsymbol{z = y^2}$

(9) $\boldsymbol{z = -x^2 - y^2}$

(10) $\boldsymbol{z = x^2 - y^2}$

(11) $\boldsymbol{y = x^2 - z^2}$

(12) $\boldsymbol{y = z^2 - x^2}$

Note: You may use **traces** to distinguish any similar graphs and identify their corresponding equations.