Solutions to Math 2400, Final Exam

December 18, 2018

PRINT YOUR NAME: ____________________

PRINT INSTRUCTOR'S NAME: ____________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50
☐	Section 002	Philip Kopel	8:00–8:50
☐	Section 003	Daniel Martin	8:00–8:50
☐	Section 004	Albert Bronstein	9:00–9:50
☐	Section 005	Mark Pullins	9:00–9:50
☐	Section 006	Xingzhou Yang	9:00–9:50
☐	Section 007	Martin Walter	10:00–10:50
☐	Section 008	Kevin Manley	10:00–10:50
☐	Section 009	Albert Bronstein	1:00–1:50
☐	Section 010	Martin Walter	1:00–1:50
☐	Section 011	Xingzhou Yang	2:00–2:50
☐	Section 012	Taylor Klotz	2:00–2:50
☐	Section 013	Xingzhou Yang	3:00–3:50
☐	Section 014	Braden Balentine	4:00–4:50
☐	Section 015	Caroline Matson	4:00–4:50

Question	Points	Score
1	12	
2	9	
3	7	
4	9	
5	6	
6	9	
7	8	
8	8	
9	8	
10	8	
11	8	
12	8	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 150 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like $\mathbf{100/7}$ or expressions like $\mathbf{\ln(3)/2}$ as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (12 points) **Note: No partial credit for this problem.**

Let $\vec{a} = \langle -1,\ 2,\ 2\rangle$, $\vec{b} = \langle 3,\ -2,\ 1\rangle$. Compute

(a) $|\vec{a}| = \boxed{3}$

> **Solution:** $|\vec{a}| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \boxed{3}$.

(b) $-2\vec{a} + 3\vec{b} = \boxed{\langle 11, -10, -1\rangle}$

> **Solution:** $-2\vec{a} + 3\vec{b} = -2\langle -1,\ 2,\ 2\rangle + 3\langle 3,\ -2,\ 1\rangle = \langle 2+9,\ -4-6,\ -4+3\rangle$
> $= \boxed{\langle 11, -10, -1\rangle}$.

(c) $\vec{a} \cdot \vec{b} = \boxed{-5}$

> **Solution:** $\vec{a} \cdot \vec{b} = \langle -1,\ 2,\ 2\rangle \cdot \langle 3,\ -2,\ 1\rangle$
> $= (-1)(3) + (2)(-2) + (2)(1) = \boxed{-5}$.

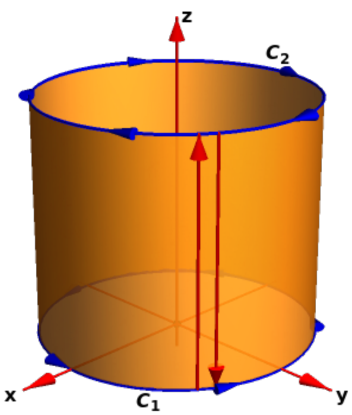
(d) $\vec{a} \times \vec{b} = \boxed{\langle 6, 7, -4\rangle}$

> **Solution:** $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 3 & -2 & 1 \end{vmatrix}$
> $= \vec{i}\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix}$
> $= (2+4)\vec{i} - (-1-6)\vec{j} + (2-6)\vec{k}$
> $= 6\vec{i} + 7\vec{j} - 4\vec{k} = \boxed{\langle 6, 7, -4\rangle}$

2. (9 points) Let $\nabla \times \vec{F} = \langle 3x, 3y, -6z \rangle$ and let C_1 and C_2 be the circles of **radius two** centered on the z-axis at $z = 0$ and $z = 5$, respectively. Calculate

$$\oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r}$$

C_1 is oriented counter-clockwise, and C_2 is oriented clockwise viewed from the positive z-axis.



Solution: Choose the surface $S_1 : z = 5$, oriented downward, with C_1 as its boundary. S_1 is the disk with radius 2 on the plane $z = 5$. S_1 can be parametrized as $\vec{r}(x, y) = \langle x, y, 5 \rangle$. Then $\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle$.

$$\mathrm{curl}\vec{F} = \langle 3x, 3y, -6z \rangle = \langle 3x,\ 3y, -30 \rangle$$
$$\mathrm{curl}\vec{F} \cdot \vec{r}_x \times \vec{r}_y = (3x)(0) + (3x)(0) + (-30)(1) = -30$$

The projection of S_1 onto xy-plane is $D_{xy} = \left\{(x, y) \middle| x^2 + y^2 \le 4\right\}$. Use the **Stokes Theorem**, and then we have (the 3rd component of $\vec{r}_x \times \vec{r}_y$ is postive, and S_1 is oriented downward, so negative sign is used in the Theorem),

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_{S_1} \mathrm{curl}\vec{F} \cdot \mathrm{d}\vec{S} = -\iint_{D_{xy}} \mathrm{curl}\vec{F} \cdot \vec{r}_x \times \vec{r}_y \, \mathrm{d}A$$
$$= -\iint_{D_{xy}} -30 \, \mathrm{d}A = 30 \iint_{D_{xy}} \mathrm{d}A = 30 \cdot \pi(2)^2 = 120\pi$$

For the 2nd line integral, we choose the surface $S_2 : z = 0$, oriented upward, with C_2 as its boundary. S_2 is the disk with radius 2 on the plane $z = 0$. S_2 can be parametrized as $\vec{r}(x, y) = \langle x, y, 0 \rangle$. Then $\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle$.

$$\mathrm{curl}\vec{F} = \langle 3x, 3y, -6z \rangle = \langle 3x,\ 3y, 0 \rangle$$
$$\mathrm{curl}\vec{F} \cdot \vec{r}_x \times \vec{r}_y = (3x)(0) + (3x)(0) + (0)(1) = 0$$

So the second line integral is zero. So we get

$$\oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} = 120\pi + 0 = \boxed{120\pi}$$

Solution 2: Vertical cut the surface (see the figure), and then we can apply the Stokes Theorem directly. Denote $\boldsymbol{S}$ the cylinder oriented outward. Parametrize $\boldsymbol{S}$ and we have
$\vec{r}(\theta, z) = \langle 2\cos\theta,\ 2\sin\theta,\ z\rangle$, $D_{\theta z} = \left\{(\theta, z)\middle| 0 \le \theta \le 2\pi, 0 \le z \le 5\right\}$.

$$\vec{r}_\theta = \langle -2\sin\theta,\ 2\cos\theta,\ 0\rangle\ ,\quad \vec{r}_z = \langle 0,\ 0,\ 1\rangle$$

$$\begin{aligned}
\vec{r}_\theta \times \vec{r}_z &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \vec{i}\begin{vmatrix} 2\cos\theta & 0 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} -2\sin\theta & 0 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} -2\sin\theta & 2\cos\theta \\ 0 & 0 \end{vmatrix} \\
&= 2\cos\theta\ \vec{i} + 2\sin\theta\ \vec{j} + 0\ \vec{k} = \langle 2\cos\theta,\ 2\sin\theta,\ 0\rangle \\
\text{curl}\vec{F} &= \langle 3x, 3y, -6z\rangle = \langle 6\cos\theta,\ 6\sin\theta,\ -6z\rangle \\
\text{curl}\vec{F}\cdot\vec{r}_\theta \times \vec{r}_z &= (2\cos\theta)(6\cos\theta) + (2\sin\theta)(6\sin\theta) + (0)(-6z) \\
&= 12(\cos^2\theta + \sin^2\theta) + 0 = 12
\end{aligned}$$

By the **Stokes Theorem**, we get (if $\theta = 0$, $\vec{r}_\theta \times \vec{r}_z = \langle 2, 0, 0\rangle$, pointing outward, so positve sign is used in the Theorem),

$$\begin{aligned}
\oint_{C_1} \vec{F}\cdot \mathrm{d}\vec{r} + \oint_{C_2} \vec{F}\cdot \mathrm{d}\vec{r} &= \iint_S \text{curl}\vec{F}\cdot \mathrm{d}\vec{S} = + \iint_{D_{\theta z}} \text{curl}\vec{F}\cdot\vec{r}_\theta \times \vec{r}_z\ \mathrm{d}A \\
&= 12\iint_{D_{\theta z}} \mathrm{d}A = 12\int_0^{2\pi}\int_0^5 \mathrm{d}z\mathrm{d}\theta = 12(2\pi)(5) \\
&= \boxed{120\pi}
\end{aligned}$$

3. (7 points) Find the distance between the two planes.

$$2x - y + 2z = 7$$
$$2x - y + 2z = 1$$

Solution: The normal directions of the two planes are the same $\vec{n} = \langle 2,\ -1,\ 2\rangle$, and they are paralle.

$$\text{distance} = \frac{|7-1|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} = \frac{6}{\sqrt{9}} = \frac{6}{3} = \boxed{2}$$

4. (9 points) Let $\boldsymbol{S}$ denote the surface given by the parameterization

$$\vec{r}(u,v) = (u^2+v^2)\vec{i} + (u^2-v^2)\vec{j} + uv\,\vec{k}, \qquad u \text{ is in } \mathbb{R}, \text{ and } v \geq 0.$$

Find an equation for the tangent plane to $\boldsymbol{S}$ at the point $P(1,\,-1,\,0)$.

Solution: To get the corresponding u, v to $P(1,\,-1,\,0)$, solve

$$u^2+v^2=1, \quad u^2-v^2=-1, \quad uv=0$$

for u and v, and we get $u=0$, $v=1$. The normal direction of the plane at $P(1,\,-1,\,0)$ is $\vec{r}_u(0,1)\times\vec{r}_v(0,1)$.

$$\vec{r}_u = \left\langle \frac{\partial}{\partial u}(u^2+v^2),\ \frac{\partial}{\partial u}(u^2-v^2),\ \frac{\partial}{\partial u}(uv) \right\rangle = \langle 2u,\ 2u,\ v\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial}{\partial v}(u^2+v^2),\ \frac{\partial}{\partial v}(u^2-v^2),\ \frac{\partial}{\partial v}(uv) \right\rangle = \langle 2v,\ -2v,\ u\rangle$$

$$\vec{n} = \vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 2u & v \\ 2v & -2v & u \end{vmatrix} = \left\langle 2u^2+2v^2,\ 2v^2-2u^2,\ -8uv\right\rangle$$

At $P(1,\,-1,\,0)$, $\vec{n} = \left\langle 2(0)^2+2(1)^2,\ 2(1)^2-2(0)^2,\ -8(0)(1)\right\rangle = \langle 2,\ 2,\ 0\rangle$. Or to be simpler, we may get $\vec{r}_u(0,1)=\langle 0,0,1\rangle$, $\vec{r}_v(0,1)=\langle 2,-2,0\rangle$, then

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{vmatrix} = \vec{i}(0+2)-\vec{j}(0-2)+\vec{k}(0-0) = \langle 2,\ 2,\ 0\rangle$$

So the equation of the tangent plane is

$$\boxed{2(x-1)+2(y+1)+0(z-0)=0} \quad \text{or} \quad \boxed{x+y=0}$$

5. (6 points) There will be **NO partial credit** awarded on the following questions – so be sure to double check your work!

(a) Suppose the amount of snow on the ground is given by the function $f(x, y) = x^4 + x^2y^2$. If you are standing at the point $P(1, 3)$, in which direction would you walk to decrease how much snow you are standing in in the fastest?

Solution: The max rate of change of f occurs at its gradient direction grad$f(1, 3)$, and min rate of change of f occurs at the negative gradient direction.

$$\text{grad} f = \nabla f = \langle f_x,\ f_y \rangle = \left\langle 4x^3 + 2xy^2,\ 2x^2y \right\rangle$$
$$\text{grad} f(1, 3) = \left\langle 4(1)^3 + 2(1)(3)^2,\ 2(1)^2(3) \right\rangle = \langle 22,\ 6 \rangle$$

So in the direction $\boxed{-\langle 22,\ 6\rangle = \langle -22,\ -6\rangle}$ you walk to decrease fastest and the amount of snow is

$$|\text{grad} f| = \sqrt{(-22)^2 + (-6)^2} = \boxed{\sqrt{520}} = \boxed{2\sqrt{130}}$$

(b) Let $\vec{F} = (ye^{xy} + y^2)\vec{\imath} + (xe^{xy} + x^2)\vec{\jmath}$. Is $\vec{F}$ conservative? If so, find the associated potential function.

Solution: Denote $P(x, y) = ye^{xy} + y^2$, $Q(x, y) = xe^{xy} + x^2$.

$$Q_x = \frac{\partial}{\partial x}\left(xe^{xy} + x^2\right) = xye^{xy} + e^{xy} + 2x$$
$$P_y = \frac{\partial}{\partial y}\left(ye^{xy} + y^2\right) = xye^{xy} + e^{xy} + 2y$$
$$Q_x - P_y = 2x - 2y$$

So $Q_x - P_y \not\equiv 0$. So $\vec{F}$ is $\boxed{\text{not conservative}}$.

(c) Suppose that C is a simple, smooth, positively-oriented curve which encloses the region R in the xy-plane. If $\displaystyle\int_C 5x\, dx + 3x\, dy = 30$, what is the area of R?

Solution: C is closed, so by Green's Theorem,

$$\int_C 5x\, dx + 3x\, dy = \iint_R \left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}5x\right] 7 = \iint_R 3\ 7 = 3\iint_R 7 = 3A(R)$$

So the area of R is $A(R) = \dfrac{30}{3} = \boxed{10}$.

6. (9 points) Use the **Method of Lagrange Multipliers** to find all extrema of

$$f(x,y) = x^2 + y^2 - 2x - 4y$$

constrained to the circle $x^2 + y^2 = 5$.

Solution: Let $g(x,y) = x^2 + y^2$. Then the constraint equation is $g(x,y) = 5$.

$$f_x = \frac{\partial}{\partial x}\left(x^2 + y^2 - 2x - 4y\right) = 2x - 2 \qquad g_x = \frac{\partial}{\partial x}\left(x^2 + y^2\right) = 2x$$

$$f_y = \frac{\partial}{\partial y}\left(x^2 + y^2 - 2x - 4y\right) = 2y - 4 \qquad g_y = \frac{\partial}{\partial y}\left(x^2 + y^2\right) = 2y$$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x - 2, 2y - 4 \rangle \qquad \nabla g = \langle g_x, g_y \rangle = \langle 2x, 2y \rangle$$

By the **Method of Lagrange Multipliers**, we solve $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 5 \end{cases}$ which is the same as

$$\begin{cases} \langle 2x - 2, 2y - 4 \rangle = \lambda \langle 2x, 2y \rangle \\ x^2 + y^2 = 5 \end{cases} \iff \begin{cases} 2x - 2 = 2\lambda x & ① \\ 2y - 4 = 2\lambda y & ② \\ x^2 + y^2 = 5 & ③ \end{cases} \iff \begin{cases} 2(1-\lambda)x = 2 & ① \\ 2(1-\lambda)y = 4 & ② \\ x^2 + y^2 = 5 & ③ \end{cases}$$

Note $\lambda \neq 1$ since otherwise it contradicts with ① and ②. From ① and ②, we get $x = \dfrac{1}{1-\lambda}$, $y = \dfrac{2}{1-\lambda}$. By ③, we have $\dfrac{5}{(1-\lambda)^2} = 5$, so $(1-\lambda)^2 = 1$. $\lambda = 0$, or 2.

When $\lambda = 0$, $x = \dfrac{1}{1-0} = 1$, $y = \dfrac{2}{1-0} = 2$,

$$f(1,2) = (1)^2 + (2)^2 - 2(1) - 4(2) = -5$$

When $\lambda = 2$, $x = \dfrac{1}{1-2} = -1$, $y = \dfrac{2}{1-2} = -2$,

$$f(-1,-2) = (-1)^2 + (-2)^2 - 2(-1) - 4(-2) = 15$$

So we get only two points $(x,y) = (-1,-2)$ and $(x,y) = (1,2)$. So the **absolute max/min values** of the function are $\boxed{15}$ and $\boxed{-5}$, respectively.

7. (8 points) Suppose that f is a differentiable function of x and y, and

$$g(s,t) = f(s^2 - 3t, 4s - t).$$

	$f(x,y)$	$g(s,t)$	$f_x(x,y)$	$f_y(x,y)$
$(-2,3)$	5	1	2	3
$(1,1)$	4	5	6	7

(a) Use the table of values to compute $g_s(1,1)$.

Solution: Let $x = s^2 - 3t$, $y = 4s - t$. When $s = 1$, $t = 1$, $x = 1^2 - 2(1) = -2$, $y = 4(1) - (1) = 3$. $\dfrac{\partial x}{\partial s} = \dfrac{\partial}{\partial s}(s^2 - 3t) = 2s$, $\dfrac{\partial y}{\partial s} = \dfrac{\partial}{\partial s}(4s - t) = 4$. So $x_s(1,1) = 2(1) = 2$, $y_s(1,1) = 4$. By the Chain Rule,

$$g_s(1,1) = f_x(-2,3)\cdot x_s(1,1) + f_y(-2,3)\cdot y_s(1,1) = (2)(2) + (3)(4) = \boxed{16}$$

(b) If $g_t(0,-1) = 3$ and $f_x(3,1) = 2$, use the table of values to compute $f_y(3,1)$.

Solution: When $s = 0$, $t = -1$, $x = (0)^2 - 3(-1) = 3$, $y = 4(0) - (-1) = 1$. By the Chain Rule,

$$\begin{aligned}
\frac{\partial}{\partial t}g(s,t) = g_t(s,t) &= \frac{\partial}{\partial t}f(x,y) = f_x(x,y)\frac{\partial x}{\partial t} + f_y(x,y)\frac{\partial y}{\partial t} \\
&= f_x(s^2-3t, 4s-t)(-3) + f_y(s^2-3t, 4s-t)(-1) \\
g_t(0,-1) &= -3f_x(3,1) - f_y(3,1) \\
3 &= -3(2) - f_y(3,1) \qquad [\text{solve it for } f_y(3,1)] \\
f_y(3,1) &= -6 - 3 = \boxed{-9}
\end{aligned}$$

8. (8 points) Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Determine if the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exists. **Justify**.

Solution: Let $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. Use polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$. Then $x^2+y^2 = r^2$.

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} = \frac{r\cos\theta\, r\sin\theta}{\sqrt{r^2}} = r\cos\theta\sin\theta$$

Since for any θ, $|\sin\theta| \leq 1$, $|\cos\theta| \leq 1$, we have

$$0 \leq |f(x,y) - 0| = |r\cos\theta\sin\theta| \leq r$$

As $(x,y) \to (0,0)$, $r = \sqrt{x^2+y^2} \to 0$, and so the lower bound and upper bound of $|f(x,y)|$ both approach 0 as $(x,y) \to (0,0)$. By the **Squeeze Theorem**, $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = \boxed{0}$. This also proves the existence of the limit.

Solution 2: Let $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. Pick a path, for example, $C : y = 0$. Then $f(x,y) = \dfrac{x^2}{\sqrt{x^2}} \to 0$, as $(x,y) \to (0,0)$. To show that 0 is the limit of the function as $(x,y) \to (0,0)$, we use the **Squeeze Theorem**.

$$0 \leq |f(x,y)| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{1}{2}\frac{2|x|\cdot|y|}{\sqrt{x^2+y^2}} \leq \frac{1}{2}\frac{|x|^2+|y|^2}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2}$$

Since $\lim_{(x,y)\to 0} 0 = 0$, and $\lim_{(x,y)\to 0} \frac{1}{2}\sqrt{x^2+y^2} = 0$, by the **Squeeze Theorem**, we get

$$\lim_{(x,y)\to 0} |f(x,y)| = 0 \Longleftrightarrow \lim_{(x,y)\to 0} f(x,y) = \boxed{0}$$

(b) Determine if f is continuous at $(0,\ 0)$. **Justify**.

Solution: The function value at $(0,0)$ is $f(0,0) = 2$, which is not equal to $\lim_{(x,y)\to 0} f(x,y) = 0$, so f is $\boxed{\text{not continuous}}$.

9. (8 points) Suppose the surface $\boldsymbol{S}$ of a small island with lizards is given by

$$z = 3e^{-x^2-y^2}$$

with $x^2 + y^2 \leq 100$ and with all distances measured in miles. The population density of the lizards at a point (x, y, z) on the island is given by

$$\rho(x, y, z) = \frac{50}{1 + x^2 + y^2}$$

lizards **per square mile**. Set up but **do not evaluate** an integral giving the total population of the lizards on the island.

$$\int_{\boxed{}}^{\boxed{}} \int_{\boxed{}}^{\boxed{}} \boxed{} \, \mathrm{d}\boxed{} \, \mathrm{d}\boxed{}$$

Solution: Let $f(x, y) = 3e^{-x^2-y^2}$. We may parametrize the surface as

$$\vec{r}(x, y) = \langle x,\ y,\ f(x, y)\rangle = \left\langle x,\ y,\ 3e^{-x^2-y^2}\right\rangle.$$

Then

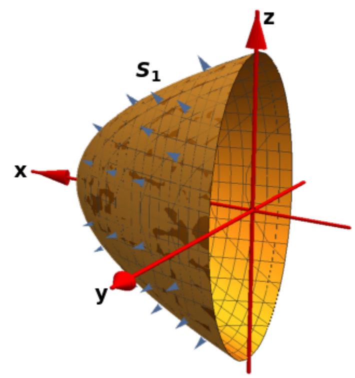
$$\begin{aligned}
\vec{r}_x \times \vec{r}_y &= \langle -f_x,\ -f_y,\ 1\rangle = \left\langle 6xe^{-x^2-y^2},\ 6ye^{-x^2-y^2},\ 1\right\rangle \\
|\vec{r}_x \times \vec{r}_y| &= \sqrt{\left(6xe^{-x^2-y^2}\right)^2 + \left(6ye^{-x^2-y^2}\right)^2 + 1^2} \\
&= \sqrt{36(x^2+y^2)e^{-2(x^2+y^2)} + 1}
\end{aligned}$$

The total popluation is

$$\begin{aligned}
\textbf{total population} &= \iint_S \rho(x, y, z)\,\mathrm{d}S = \iint_S \frac{50}{1 + x^2 + y^2}\mathrm{d}S \\
&= \iint_{D_{xy}} \frac{50}{1 + x^2 + y^2}\,|\vec{r}_x \times \vec{r}_y|\,\mathrm{d}A \\
&= \iint_{D_{xy}} \frac{50\sqrt{36(x^2+y^2)e^{-2(x^2+y^2)} + 1}}{1 + x^2 + y^2}\mathrm{d}A \\
&= \boxed{\int_{-10}^{10} \int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}} \frac{50\sqrt{36(x^2+y^2)e^{-2(x^2+y^2)} + 1}}{1 + x^2 + y^2}\,\mathrm{d}y\,\mathrm{d}x} \\
&\overset{\text{or}}{=} \boxed{\int_0^{2\pi} \int_0^{10} \frac{50r\sqrt{36r^2e^{-2r^2} + 1}}{1 + r^2}\,\mathrm{d}r\,\mathrm{d}\theta}
\end{aligned}$$

10. (8 points) Let $\boldsymbol{S_1}$ be the part of the paraboloid $\boldsymbol{x = 1 - y^2 - z^2}$, oriented outward with $\boldsymbol{x \geq 0}$, and $\boldsymbol{E}$ be the solid enclosed by $\boldsymbol{S_1}$ and the $\boldsymbol{yz}$-plane.
Let $\boldsymbol{S}$ be the boundary (closed surface) of $\boldsymbol{E}$, and

$$\vec{F} = \left\langle 2 + x\left(y^2 + z^2\right),\, x^3 + y^3,\, \sin(x^2) + z^3 \right\rangle.$$



(a) Use the **Divergence Theorem** to evaluate the flux through the surface $\boldsymbol{S}$, oriented outward.

$$\iint_S \vec{F} \cdot d\vec{S}$$

Solution: $\boldsymbol{S}$ is a closed surface with positive orientation.

$$\begin{aligned}\operatorname{div} \vec{F} &= \frac{\partial}{\partial x}\left(2 + x\left(y^2+z^2\right)\right) + \frac{\partial}{\partial y}\left(x^3+y^3\right) + \frac{\partial}{\partial z}\left(\sin(x^2)+z^3\right) \\ &= y^2 + z^2 + 3y^2 + 3z^2 = 4(y^2+z^2)\end{aligned}$$

By the Divergence Theorem,

$$\begin{aligned}\iint_S \vec{F} \cdot \mathrm{d}\vec{S} &= + \iiint_E 4(y^2+z^2)\,\mathrm{d}V \quad \left(\begin{array}{l}\text{By cylindrical coordinates} \\ x = x,\ y = r\cos\theta,\ z = r\sin\theta\end{array}\right) \\ &= 4\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r^2 \cdot r\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}\theta = 4\int_0^{2\pi}\int_0^1 r^3 x\Big|_{x=0}^{x=1-r^2}\mathrm{d}r\,\mathrm{d}\theta \\ &= 8\pi\int_0^1 r^3(1-r^2)\mathrm{d}r = 8\pi\int_0^1 (r^3 - r^5)\mathrm{d}r = 8\pi\left(\frac{r^4}{4} - \frac{r^6}{6}\right)\Big|_0^1 \\ &= 8\pi\left(\frac{1}{4} - \frac{1}{6}\right) = \boxed{\frac{2\pi}{3}}\end{aligned}$$

(b) Evaluate the flux $\displaystyle\iint_{S_1} \boldsymbol{F} \cdot d\vec{S}$.

Hint: $\boldsymbol{S_1}$ is **NOT** a closed surface, and you may use the result in (a).

Solution: Let $\boldsymbol{S_2}$ be the surface, oriented right, which makes $\boldsymbol{S_1 + S_2 = S}$. Then its equation is $\boldsymbol{x = 0}$, and it can be parametrized as $\vec{r}(y,z) = \langle 0, y, z\rangle$, where $y^2 + z^2 \leq 1$. So $\vec{F} = \langle 2,\, y^3,\, z^3\rangle$, and $\vec{r}_y \times \vec{r}_z = \langle 1, 0, 0\rangle$, pointing to the positive $\boldsymbol{x}$-axis. So

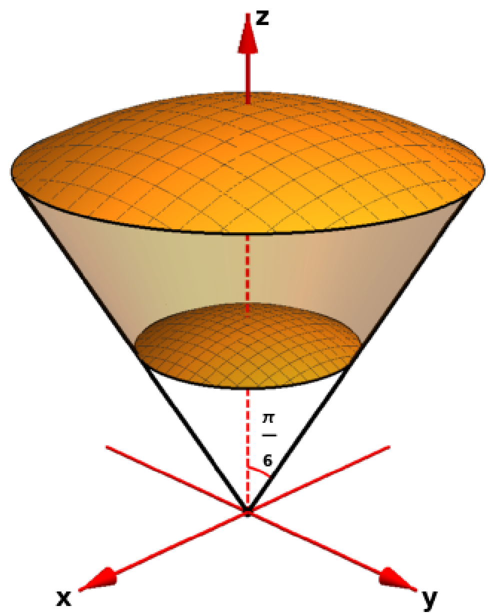
$$\iint_{S_2} \vec{F} \cdot \mathrm{d}\vec{S} = -\iint_D \vec{F} \cdot (\vec{r}_y \times \vec{r}_z)\,\mathrm{d}A = -\iint_D 2\,\mathrm{d}A = -2\pi(1)^2 = -2\pi$$

So the flux is

$$\iint_{S_1} \vec{F} \cdot \mathrm{d}\vec{S} = \iint_S \vec{F} \cdot \mathrm{d}\vec{S} - \iint_{S_2} \vec{F} \cdot \mathrm{d}\vec{S} = \frac{2\pi}{3} - (-2\pi) = \boxed{\frac{8\pi}{3}}$$

11. (8 points) Find the volume of the solid bounded by

$$\boldsymbol{x^2 + y^2 + z^2 = 4,}$$
$$\boldsymbol{x^2 + y^2 + z^2 = 1,} \quad \text{and}$$
$$\boldsymbol{z = \sqrt{3}\sqrt{x^2 + y^2}.}$$



Solution: The volume of the solid is $\iiint_E \mathrm{d}V$, and it can be solved by spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$.

$$\begin{aligned}
\text{Volume} &= \iiint_E \mathrm{d}V = \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \int_1^2 \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \int_0^{\frac{\pi}{6}} \sin\phi \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta \int_1^2 \rho^2 \\
&= -\cos\phi \Big|_0^{\frac{\pi}{6}} (2\pi) \cdot \frac{\rho^3}{3} \Big| \int_1^2 = \left(1 - \frac{\sqrt{3}}{2}\right)(2\pi)\frac{1}{3}(8-1) \\
&= \boxed{\frac{7}{3}\left(2 - \sqrt{3}\right)\pi}
\end{aligned}$$

12. (8 points) Below are a series of statements concerning gradient, curl and divergence. Assume that $\boldsymbol{f}$, $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ are scalar functions, $\vec{\boldsymbol{F}}$ is a vector field in $\mathbb{R}^3$. If all of the second order partial derivatives exist and are continuous, circle the answer that best describes each statement.

(a) $\mathbf{curl}(\mathbf{grad}\,\boldsymbol{f}) = \vec{\mathbf{0}}$.

(A) $\boxed{\text{Always true}}$

(B) Sometimes true

(C) Never true

(b) $\mathbf{div}(\mathbf{curl}\,\vec{\boldsymbol{F}}) = \mathbf{0}$.

(A) $\boxed{\text{Always true}}$

(B) Sometimes true

(C) Never true

(c) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \boldsymbol{f}) = \mathbf{0}$.

(A) Always true

(B) $\boxed{\text{Sometimes true}}$

(C) Never true

(d) If $\vec{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \langle \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{y}),\ \boldsymbol{Q}(\boldsymbol{y}),\ \boldsymbol{R}(\boldsymbol{x}, \boldsymbol{z}) \rangle$, then $\mathbf{curl}\,\vec{\boldsymbol{F}}$ is orthogonal to the $\boldsymbol{x}$-axis.

(A) $\boxed{\text{Always true}}$

(B) Sometimes true

(C) Never true