Math 2400, Midterm 3

December 3, 2018

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50
☐	Section 002	Philip Kopel	8:00–8:50
☐	Section 003	Daniel Martin	8:00–8:50
☐	Section 004	Albert Bronstein	9:00–9:50
☐	Section 005	Mark Pullins	9:00–9:50
☐	Section 006	Xingzhou Yang	9:00–9:50
☐	Section 007	Martin Walter	10:00–10:50
☐	Section 008	Kevin Manley	10:00–10:50
☐	Section 009	Albert Bronstein	1:00–1:50
☐	Section 010	Martin Walter	1:00–1:50
☐	Section 011	Xingzhou Yang	2:00–2:50
☐	Section 012	Taylor Klotz	2:00–2:50
☐	Section 013	Xingzhou Yang	3:00–3:50
☐	Section 014	Braden Balentine	4:00–4:50
☐	Section 015	Caroline Matson	4:00–4:50

Question	Points	Score
1	12	
2	14	
3	12	
4	14	
5	14	
6	3	
7	3	
8	14	
9	14	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like **100/7** or expressions like **ln(3)/2** as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (12 points) Match the vector fields $\vec{F}$ with the plots below.

(1) $\vec{F} = \langle x, y \rangle$ $\underline{(C)}$

(2) $\vec{F} = \langle y, x \rangle$ $\underline{(D)}$

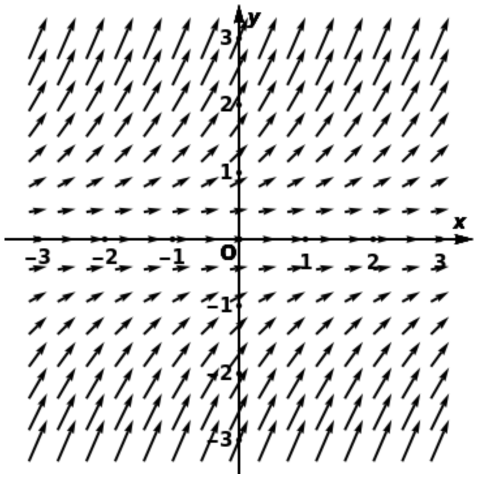
(3) $\vec{F} = \langle -x, y \rangle$ $\underline{(E)}$

(4) $\vec{F} = \langle x, x - y \rangle$ $\underline{(F)}$

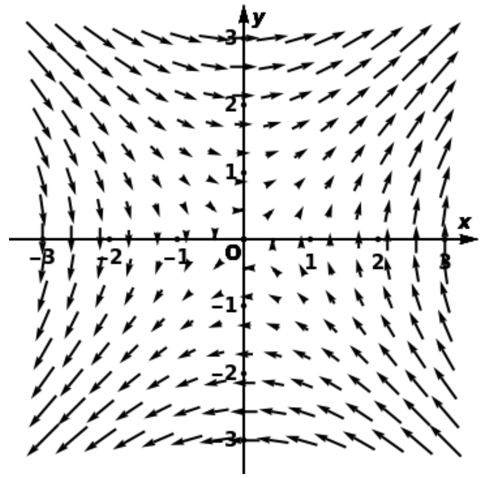
(5) $\vec{F} = \langle 1, x \sin y \rangle$ $\underline{(B)}$

(6) $\vec{F} = \langle 1, \ln(1 + y^2) \rangle$ $\underline{(A)}$

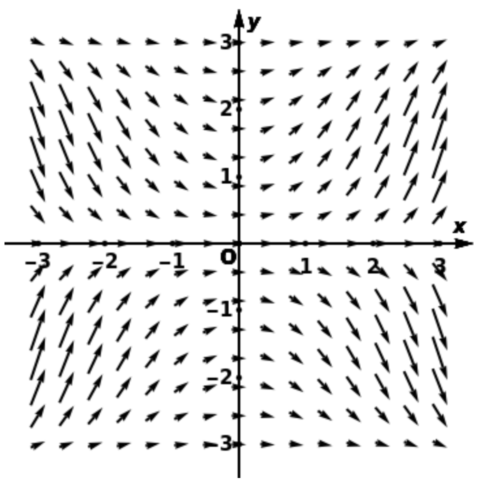
(A)



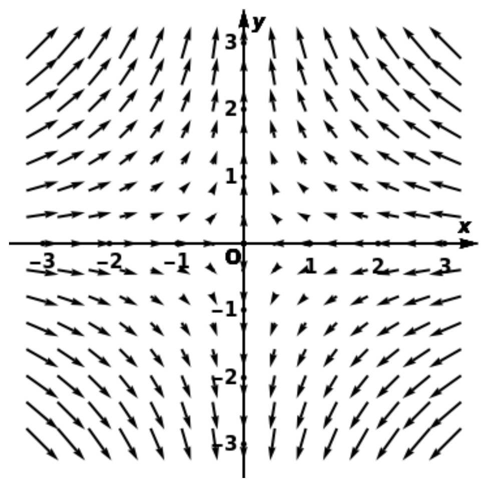
(D)



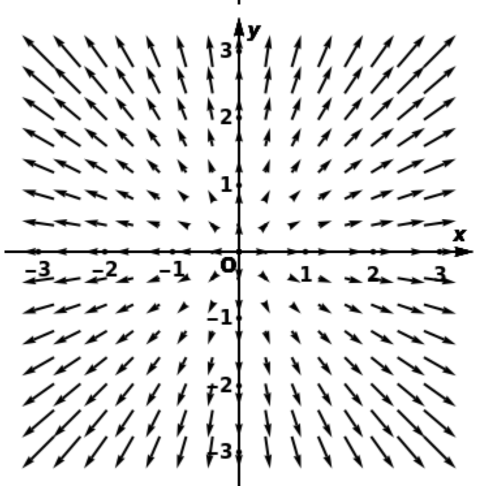
(B)



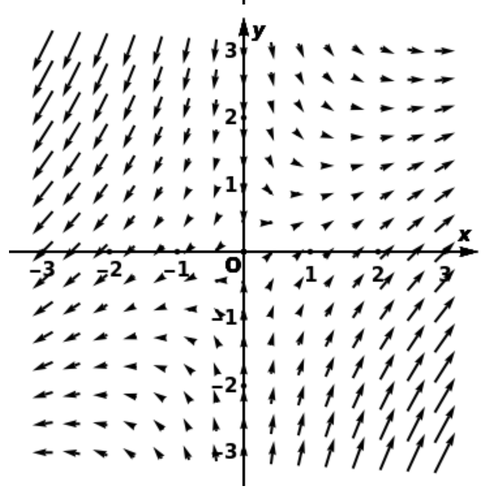
(E)



(C)



(F)



2. (14 points) Consider the vector field $\vec{F} = (2xy + e^x)\vec{i} + x^2\vec{j}$.

(a) Determine if $\vec{F}$ is a conservative vector field. If so, find a potential function of $\vec{F}$. If not, explain how you can tell.

Solution: Let $P(x,y) = 2xy + e^x$, $Q(x,y) = x^2$. Then $Q_x = \frac{\partial}{\partial x}(x^2) = 2x$, $P_y = \frac{\partial}{\partial y}(2xy + e^x) = 2x$. So $Q_x = P_y$ for all $(x,y) \in \mathbb{R}^2$. Hence $\vec{F}$ is conservative.

So there exist f such that $\nabla f = \vec{F}$. That is

$$\langle f_x, f_y \rangle = \langle 2xy + e^x, x^2 \rangle \Longleftrightarrow \begin{cases} f_x = 2xy + e^x & ① \\ f_y = x^2 & ② \end{cases}$$

Integrate both sides of ① with respect to x and we get

$$f(x,y) = \int f_x\, dx = \int (2xy + e^x)\, dx = x^2 y + e^x + C(y) \qquad ③$$

Plug it into ② and we have

$$x^2 \overset{②}{=} f_y \overset{③}{=} \frac{\partial}{\partial y}\left(x^2 y + e^x + C(y)\right) = x^2 + C'(y) \Leftrightarrow C'(y) = 0 \Leftrightarrow C(y) = K$$

So we get a potential function f

$$\boxed{f(x,y) = x^2 y + e^x + K, \quad \text{where } K \text{ can be any real numbers.}}$$

It is easy to check that $\nabla f = \langle 2xy + e^x, x^2 \rangle = \vec{F}$.

(b) Let C be the curve given by

$$C\colon \vec{r}(t) = \ln(1 + t^{2018})\vec{i} + \ln(1 + t^2 + t^4 + t^6 + t^8)\vec{j}, \quad 0 \le t \le 1$$

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

Solution: The initial and terminal points, denoted by A and B, respectively, of C are $\vec{r}(0) = \langle 0, 0 \rangle$ and $\vec{r}(1) = \langle \ln 2, \ln 5 \rangle$. From (a), $\vec{F}$ is conservative. $\vec{F}$ has continuous partial derivatives on $\mathbb{R}^2$. We choose $f(x,y) = x^2 y + e^x$ $(K = 0)$, and use the Fundamental Theorem of Line Integrals.

$$\int_C \vec{F} \cdot d\vec{r} = f(\ln 2, \ln 5) - f(0,0) = \left[e^{\ln 2} + (\ln 2)^2(\ln 5)\right] - \left[e^0 + (0)^2(0)\right]$$
$$= 2 + (\ln 2)^2 \ln 5 - 1 = \boxed{1 + (\ln 2)^2 \ln 5}$$

Solution 2: Since $\vec{F}$ is conservative, $\int_C \vec{F} \cdot d\vec{r}$ is independent of path, and it equals $\int_{\overrightarrow{AB}} \vec{F} \cdot d\vec{r}$, where $\overrightarrow{AB}$ is the line segment from $A(0,0)$ to $B(\ln 2, \ln 5)$. Its equation is $\vec{r}(t) = (1-t)\langle 0, 0 \rangle + t\langle \ln 2, \ln 5 \rangle = \langle t \ln 2, t \ln 5 \rangle$. $\vec{r}\,'(t) = \langle \ln 2, \ln 5 \rangle$. Along

C, $\vec{F} = \left\langle 2(t\ln 2)(t\ln 5) + e^{t\ln 2}, (t\ln 2)^2 \right\rangle = \langle 2\ln 2\ln 5\; t^2 + 2^t,\; (\ln 2)^2 t^2\rangle$

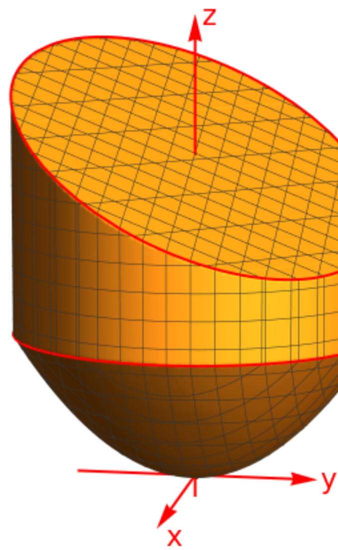
$$\vec{F}\cdot d\vec{r} = \vec{F}\cdot\vec{r}\,'(t)\,dt = \left[2(\ln 2)^2\ln 5\; t^2 + 2^t\ln 2 + (\ln 2)^2\ln 5\; t^2\right]dt$$

$$= \left[3(\ln 2)^2\ln 5\; t^2 + 2^t\ln 2\right]dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \left[3(\ln 2)^2\ln 5\; t^2 + 2^t\ln 2\right]dt = \left[(\ln 2)^2\ln 5\; t^3 + 2^t\right]\Big|_0^1$$

$$= \boxed{1 + (\ln 2)^2\ln 5}$$

3. (12 points) A solid E is bounded by the plane $x + y + z = 9$ and the paraboloid $z = x^2 + y^2$ within the cylinder $x^2 + y^2 = 4$. The density of the mass is $\rho(x, y, z) = \sqrt{x^2 + y^2}$.
Set up the integral for the **mass** of the solid E in **cylindrical coordinates**.
Do **not** evaluate the integral.



Solution: The mass of the solid is $m = \iiint_E \rho(x, y, z)\,dV$. In cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, $dV = r\,dz\,dr\,d\theta$, and the equations of the surfaces are plane: $r\cos\theta + r\sin\theta + z = 9$, paraboloid: $z = r^2$, cylinder: $r = 2$.
The projection of the solid onto xy-plane is a circle: $D = \{(x, y)|x^2 + y^2 \le 4\}$, and in cylindrical coordinates, the region becomes $\{(r, \theta)|0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$.

$$m = \iiint_E \rho(x, y, z)\,dV = \boxed{\int_0^{2\pi}\int_0^2\int_{r^2}^{9 - r\cos\theta - r\sin\theta} r\cdot r\,dz\,dr\,d\theta}$$

Solution 2:

$$m = \boxed{\int_0^{2\pi}\int_0^4\int_0^{\sqrt{z}} r\cdot r\,dr\,dz\,d\theta + \int_0^{2\pi}\int_0^2\int_4^{9 - r\cos\theta - r\sin\theta} r\cdot r\,dz\,dr\,d\theta}$$

4. (14 points) Determine the surface area of $\boldsymbol{S}$, where $\boldsymbol{S}$ is the parametric surface

$$\vec{r}(u,v) = \langle u,\ v,\ 4-u^2-v^2\rangle$$

for $u^2+v^2 \leq 2$.

Solution: Let $D = \{(u,v)|u^2+v^2 \leq 2\}$.

$$\vec{r}_u = \frac{\partial}{\partial u}\langle u, v, 4-u^2-v^2\rangle = \langle 1, 0, -2u\rangle$$

$$\vec{r}_v = \frac{\partial}{\partial v}\langle u, v, 4-u^2-v^2\rangle = \langle 0, 1, -2v\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -2u \\ 1 & -2v \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & -2u \\ 0 & -2v \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 2u\,\vec{i} + 2v\,\vec{j} + \vec{k} = \langle 2u,\ 2v,\ 1\rangle$$

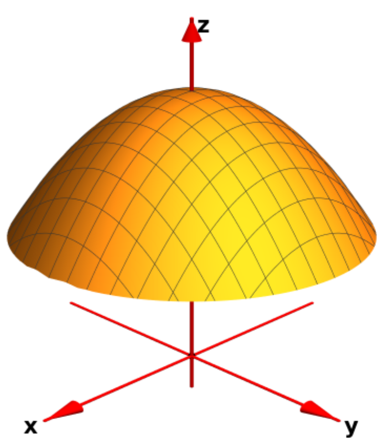
$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(2u)^2+(2v)^2+(1)^2} = \sqrt{4(u^2+v^2)+1}$$

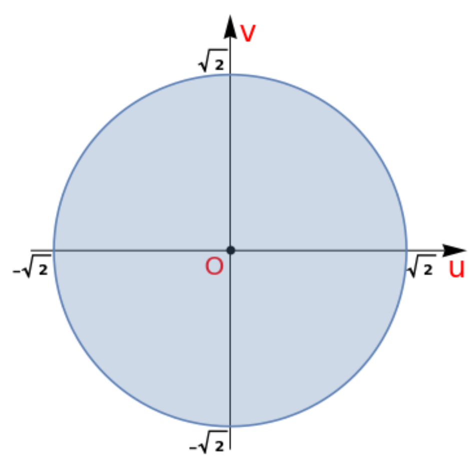
$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA = \iint_D \sqrt{4(u^2+v^2)+1}\,dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \sqrt{4r^2+1}\,r\,dr$$

$$= 2\pi \int_0^{\sqrt{2}} \sqrt{4r^2+1}\,r\,dr \overset{u=4r^2+1}{\underset{du=8r\,dr}{=\!=\!=}} 2\pi \int_1^9 \sqrt{u}\cdot\frac{1}{8}du$$

$$= \frac{\pi}{4}\int_1^9 \sqrt{u}\,du = \frac{\pi}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_1^9 = \frac{\pi}{6}\left(9^{\frac{3}{2}}-1\right) = \boxed{\frac{13\pi}{3}}$$

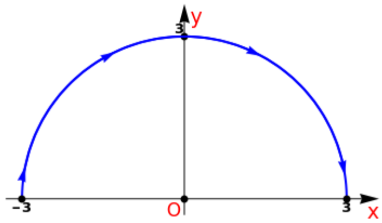




5. (14 points) Consider the curve C given by the half circle of radius 3 centered at the origin, traversed **CLOCKWISE** from $(-3, 0)$ to $(3, 0)$.

(a) Find a parameterization of C. Include bounds.

Solution: Let $x = 3\sin\theta$, $y = 3\cos\theta$. $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.



Or: $x = -3\sin\theta$, $y = -3\cos\theta$. $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.
Or: $x = 3\cos\theta$, $y = -3\sin\theta$. $\pi \le \theta \le 2\pi$.
Or: $x = -3\cos\theta$, $y = 3\sin\theta$. $0 \le \theta \le \pi$.

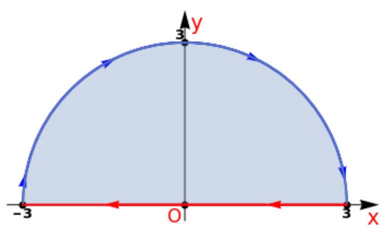
(b) Let $\vec{F} = \langle 4y, -4x \rangle$. Evaluate the integral $\int_C \vec{F} \cdot d\vec{r}$

Solution: C in vector form is $\vec{r}(\theta) = \langle 3\sin\theta, 3\cos\theta \rangle$. $\vec{r}\,'(\theta) = \langle 3\cos\theta, -3\sin\theta \rangle$.
$\vec{F}$ along C is $\vec{F} = \langle 4 \cdot 3\cos\theta, -4 \cdot 3\sin\theta \rangle = \langle 12\cos\theta, -12\sin\theta \rangle$.

$$\begin{aligned}\vec{F} \cdot d\vec{r} = \vec{F} \cdot \vec{r}\,'(\theta)\, d\theta &= \langle 12\cos\theta, -12\sin\theta \rangle \cdot \langle 3\cos\theta, -3\sin\theta \rangle\, d\theta \\ &= (12\cos\theta)(3\cos\theta) + (-12\sin\theta)(-3\sin\theta) = 36(\cos^2\theta + \sin^2\theta) \\ &= 36\end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 36 d\theta = \boxed{36\pi}$$

Solution 2: If we add a path C_1, a line segment from $(3, 0)$ to $(-3, 0)$, then the path C and C_1 enclose a region D with boundary C and C_1 with clockwise orientation.



Use Green's Theorem,

$$\int_{C+C_1} \vec{F} \cdot d\vec{r} = \int_{C+C_1} 4y\, dx - 4x\, dy = -\iint_D \left(\frac{\partial}{\partial x}(-4x) + \frac{\partial}{\partial y}(4y) \right) dA$$
$$= -\iint_D (-8) dA = 8A(D) = 8 \cdot \frac{1}{2} \cdot \pi(3)^2 = 36\pi$$

Along the path from $(3, 0)$ to $(-3, 0)$, $y = 0$, and x from 3 to -3.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} 4y\, dx - 4x\, dy = \int_3^{-3} 4(0)dx - 4x\, d(0) = 0$$
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C+C_1} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r} = 36\pi - 0 = \boxed{36\pi}$$

6. (3 points) **No work is required** for this problem.

Select the integral that is **ALWAYS** equivalent to $\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x,y,z)\, dz\, dx\, dy$.

(A) $\int_0^2 \int_{x^3}^8 \int_0^{y^2} f(x,y,z)\, dz\, dy\, dx$

(B) $\int_0^2 \int_{x^2}^8 \int_0^{y^3} f(x,y,z)\, dz\, dy\, dx$

(C) $\boxed{\int_0^8 \int_{\sqrt[3]{x}}^2 \int_0^{y^2} f(x,y,z)\, dz\, dy\, dx}$

(D) $\int_0^8 \int_{\sqrt{x}}^2 \int_0^{y^3} f(x,y,z)\, dz\, dy\, dx$

(E) $\int_0^8 \int_0^{\sqrt[3]{x}} \int_0^{y^2} f(x,y,z)\, dz\, dy\, dx$

(F) $\int_0^8 \int_0^{\sqrt{x}} \int_0^{y^3} f(x,y,z)\, dz\, dy\, dx$

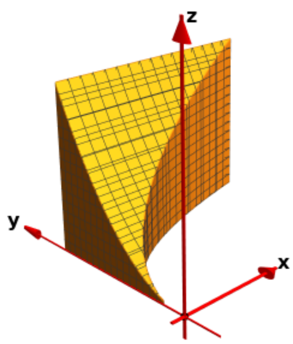
(G) $\int_0^2 \int_2^{\sqrt[3]{x}} \int_0^{y^2} f(x,y,z)\, dz\, dy\, dx$

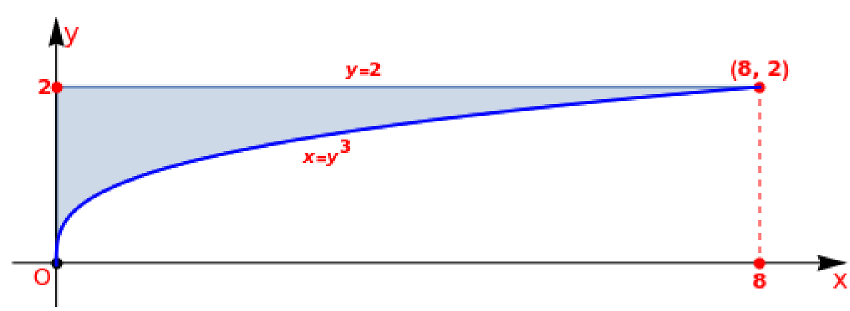
(H) $\int_0^2 \int_2^{\sqrt{x}} \int_0^{y^3} f(x,y,z)\, dz\, dy\, dx$

$$E = \{(x,y,z) \mid 0 \le z \le y^2,\ (x,y) \in D_{xy}\}$$

$$D_{xy} = \{(x,y) \mid 0 \le y \le 2,\ 0 \le x \le y^3\} \qquad \text{(type 2 region)}$$

$$= \{(x,y) \mid 0 \le x \le 8,\ \sqrt[3]{x} \le y \le 2\} \qquad \text{(type 1 region)}$$





7. (3 points) **No work is required** for this problem.
Select the integral in **spherical coordinates** that is equivalent to

$$\int_{-\frac{\sqrt{2}}{2}}^{0}\int_{-x}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z^2\,dz\,dy\,dx + \int_{0}^{\frac{\sqrt{2}}{2}}\int_{x}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z^2\,dz\,dy\,dx$$

(A) $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\frac{\sqrt{2}}{2}} \rho^4\cos^2\phi\,\sin\phi\,d\rho\,d\theta\,d\phi$

(B) $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\frac{\sqrt{2}}{2}} \rho^3\cos^2\phi\,\sin\phi\,d\rho\,d\theta\,d\phi$

(C) $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{1} \rho^2\cos^2\phi\,d\rho\,d\theta\,d\phi$

(D) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho^4\cos^2\phi\,\sin\phi\,d\rho\,d\theta\,d\phi$

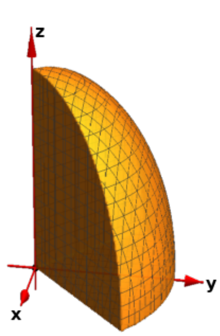
(E) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{1} \rho^2\sin^2\phi\,\cos\phi\,d\rho\,d\theta\,d\phi$

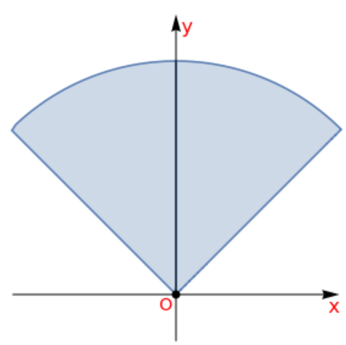
(F) $\boxed{\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{1} \rho^4\cos^2\phi\,\sin\phi\,d\rho\,d\theta\,d\phi}$

(G) $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{1} \rho^3\cos^2\phi\,\sin\phi\,d\rho\,d\theta\,d\phi$

(H) $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{1} \rho^2\sin^2\phi\,\cos\phi\,d\rho\,d\theta\,d\phi$

$$E = \left\{(x,y,z)\middle|0 \le z \le \sqrt{1-x^2-y^2},\ -x \le y \le \sqrt{1-x^2},\ -\frac{\sqrt{2}}{2} \le x \le 0\right\}$$
$$\cup\left\{(x,y,z)\middle|0 \le z \le \sqrt{1-x^2-y^2},\ x \le y \le \sqrt{1-x^2},\ 0 \le x \le \frac{\sqrt{2}}{2}\right\}$$
$$E = \left\{(\rho,\theta,\phi)\middle|0 \le \rho \le 1,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ 0 \le \phi \le \frac{\pi}{2}\right\}$$ **(in spherical coordinates)**



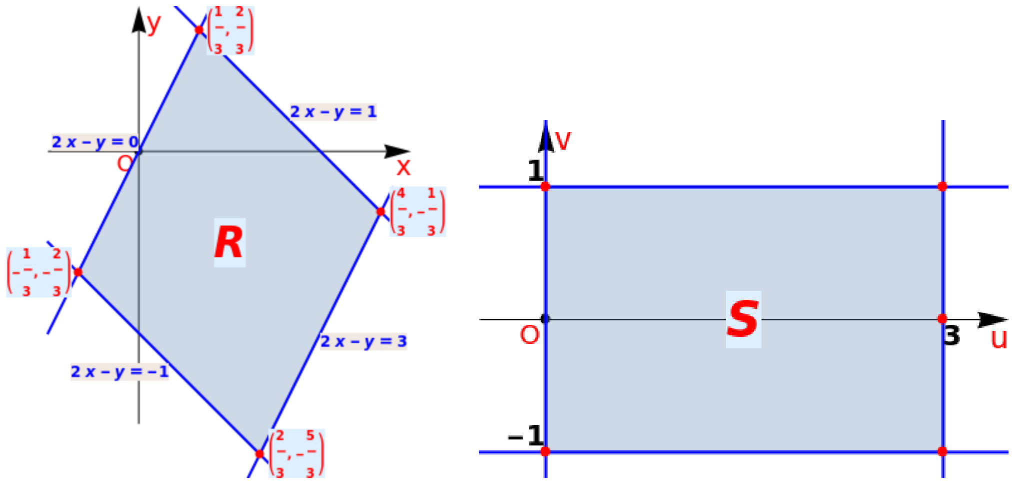


8. (14 points) Let $\boldsymbol{R}$ be the region in the $\boldsymbol{xy}$-plane bounded by the lines

$$\boldsymbol{2x - y = 0, \quad 2x - y = 3, \quad x + y = -1,} \text{ and } \boldsymbol{x + y = 1.}$$

(a) Find a transformation that maps $\boldsymbol{R}$ onto a rectangle $\boldsymbol{S}$ in the $\boldsymbol{uv}$-plane. Sketch the rectangle $\boldsymbol{S}$ in the $\boldsymbol{uv}$-plane.

Solution: Let $\boldsymbol{T}$: $\boldsymbol{u = 2x - y}$, $\boldsymbol{v = x + y}$. Then the transformation $\boldsymbol{T}$ mages $\boldsymbol{R}$ onto a rectangle $\boldsymbol{S = \{(u, v) | 0 \le u \le 3, \ -1 \le v \le 1\}}$.



(b) Use the result in (a) to find an appropriate Jacobian, and use it to evaluate the integral

$$\iint_{\boldsymbol{R}} \boldsymbol{(2x - y)^3 \, dA.}$$

Solution: From (a), $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x - y) = 2$, $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x - y) = -1$, $\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x + y) = 1$, $\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x + y) = 1$.

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = (2)(1) - (-1)(1) = 3$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1} = \frac{1}{3}$$

$$\iint_R (2x - y)^3 \, dA = \iint_S u^3 \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dA' = \int_{-1}^{1} \int_0^3 u^3 \left|\frac{1}{3}\right| du \, dv$$

$$= \frac{1}{3} \int_{-1}^{1} dv \int_0^3 u^3 \, du = \frac{1}{3} v \Big|_{-1}^{1} \cdot \frac{u^4}{4}\Big|_0^3 = \frac{2}{3} \cdot \frac{1}{4}(3^4 - 0^4) = \boxed{\frac{27}{2}}$$

Note: If we solve $u = 2x - y$, $v = x + y$ for x, y, we get $x = \frac{u+v}{3}$, $y = -\frac{u}{3} + \frac{2v}{3}$ and the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ can be computed directly,

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) - \left(\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{3}{9} = \frac{1}{3}.$$

9. (14 points) Consider the vector field

$$\vec{F} = \left(2y - e^{\sin(x)}\right)\vec{i} + \left(\ln(1+y^2)^2 + 8x\right)\vec{j}$$

and let C denote a curve which traverses the boundary of the rectangle with vertices $(0,0)$, $(0,5)$, $(5,0)$ and $(5,5)$ **exactly once** in the **counterclockwise** direction.
Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$.

Solution: Let $P(x,y) = 2y - e^{\sin(x)}$, $Q(x,y) = \ln(1+y^2)^2 + 8x$. Then $Q_x = \frac{\partial}{\partial x}\left(\ln(1+y^2)^2 + 8x\right) = 8$, $P_y = \frac{\partial}{\partial y}\left(2y - e^{\sin(x)}\right) = 2$, and Q_x and P_y are continuous when $(x,y) \in D = \left\{(x,y)\middle|0 \le x \le 5,\ 0 \le y \le 5\right\}$, the given rectangle region. By Green's Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA = \iint_D (8-2)\,dA = \iint_D 6\,dA$$
$$= 6\iint_D dA = 6A(D) = 6\cdot(5-0)(5-0) = \boxed{150}$$