Math 2400, Midterm 2

October 22, 2018

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50
☐	Section 002	Philip Kopel	8:00–8:50
☐	Section 003	Daniel Martin	8:00–8:50
☐	Section 004	Albert Bronstein	9:00–9:50
☐	Section 005	Mark Pullins	9:00–9:50
☐	Section 006	Xingzhou Yang	9:00–9:50
☐	Section 007	Martin Walter	10:00–10:50
☐	Section 008	Kevin Manley	10:00–10:50
☐	Section 009	Albert Bronstein	1:00–1:50
☐	Section 010	Martin Walter	1:00–1:50
☐	Section 011	Xingzhou Yang	2:00–2:50
☐	Section 012	Taylor Klotz	2:00–2:50
☐	Section 013	Xingzhou Yang	3:00–3:50
☐	Section 014	Braden Balentine	4:00–4:50
☐	Section 015	Caroline Matson	4:00–4:50

Question	Points	Score
1	10	
2	6	
3	10	
4	8	
5	12	
6	12	
7	12	
8	10	
9	10	
10	10	
Total:	100	

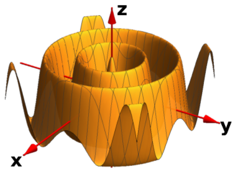
Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

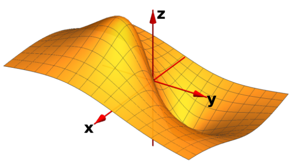
- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like **100/7** or expressions like **ln(3)/2** as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (10 points) Match each 3D surface with one of the contour plots, and one of the equations.

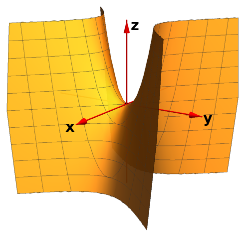
(a) **(4), (E)**



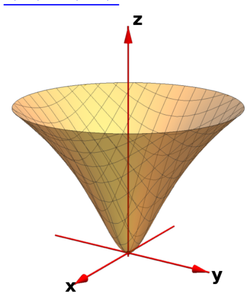
(b) **(5), (A)**



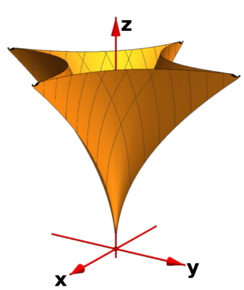
(c) **(2), (C)**



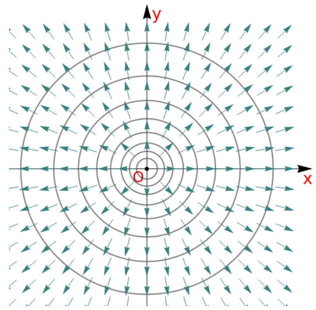
(d) **(1), (B)**



(e) **(3), (D)**



(1)



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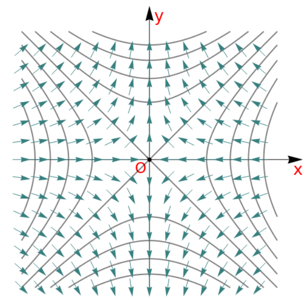

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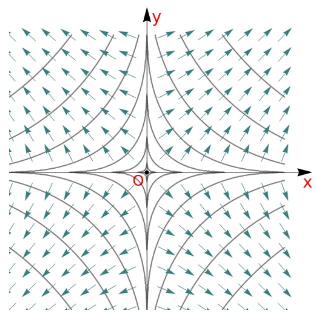

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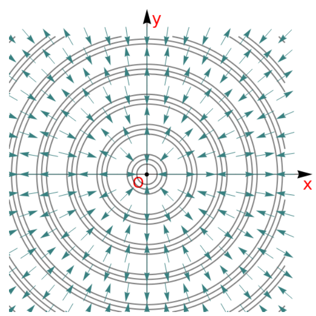
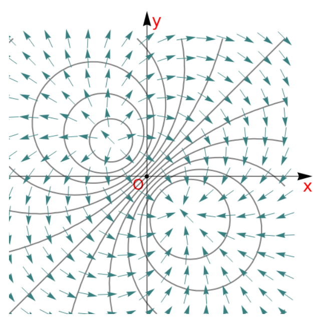


(5)



(A) $z = \dfrac{x-y}{1+x^2+y^2}$

Let $z = k$, e.g., $k = \frac{1}{2}$. Then the contour line eqn is $\frac{1}{2} = \frac{x-y}{1+x^2+y^2}$, which is equivalent to $1 + x^2 + y^2 = 2(x - y)$ $(x-1)^2 + (y+1)^2 = 1$

(B) $z = \ln(1 + x^2 + y^2)$

Let $z = k$. Then the contour line equation is $x^2 + y^2 = e^k - 1$.

(C) $z = y^2 - x^2$

Let $z = k$. Then the contour line equation is $y^2 - x^2 = k$, which is a hyperbola, or lines ($y = \pm x$ as $k = 0$).

(D) $z = \sqrt{|x|} + \sqrt{|y|}$

Let $z = k$. Then the contour line equation is $\sqrt{|x|} + \sqrt{|y|} = k$, [see figures (e) and (3)]

(E) $z = \cos(\sqrt{x^2 + y^2})$

Let $z = k$. Then the contour line equation is $x^2 + y^2 = (\arccos k + 2n\pi)^2$, $n \in \mathbb{Z}$, which is a circle. [see figures (a) and (4)]

Note: The gradient of f ∇f can also be used to determine the corresponding contour map (see arrows).

2. **Note: No partial credit for this problem.**

Consider function $\boldsymbol{f(x,y) = e^x + \cos(xy) + y^7}$. Compute

(a) (2 points) $\boldsymbol{f_x(x,y) = \underline{e^x - y\ \sin(xy)}}$.

Solution: $\boldsymbol{f_x = \dfrac{\partial}{\partial x}\left(e^x + \cos(xy) + y^7\right) = e^x - y\ \sin(xy)}$

(b) (2 points) $\boldsymbol{f_y(x,y) = \underline{-x\ \sin(xy) + 7y^6}}$.

Solution: $\boldsymbol{f_y = \dfrac{\partial}{\partial y}\left(e^x + \cos(xy) + y^7\right) = -x\ \sin(xy) + 7y^6}$

(c) (2 points) $\boldsymbol{\nabla f(0,1) = \underline{\langle 1, 7\rangle}}$.

Solution:

$$\boldsymbol{\nabla f = \langle f_x, f_y\rangle = \left\langle e^x - y\ \sin(xy), -x\ \sin(xy) + 7y^6\right\rangle}$$
$$\boldsymbol{\nabla f(0,1) = \left\langle e^0 - (1)\sin\left((0)(1)\right), -(0)\sin((0)(1)) + 7(1)^6\right\rangle = \langle 1, 7\rangle}$$

3. **Note: No partial credit for this problem.**

Consider the surface $\boldsymbol{S}$: $\boldsymbol{z = f(x, y) = x^3 - xy + 3}$.

(a) (5 points) The directional derivative of $\boldsymbol{f}$ at $\boldsymbol{(0, 1)}$ in the direction of $\vec{\boldsymbol{v}} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$ is

$\underline{-\frac{3}{5}}$.

Solution: $\vec{v}$ **is a unit vector since** $\sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$.

$$f_x = \frac{\partial}{\partial x}\left(x^3 - xy + 3\right) = 3x^2 - y$$
$$f_y = \frac{\partial}{\partial y}\left(x^3 - xy + 3\right) = -x$$
$$\nabla f = \langle f_x, f_y \rangle = \left\langle 3x^2 - y, -x \right\rangle$$
$$\nabla f(0, 1) = \left\langle 3(0)^2 - (1),\ -(0) \right\rangle = \langle -1, 0 \rangle$$
$$D_{\vec{v}} f(0, 1) = \nabla f(0, 1) \cdot \vec{v} = (-1)\left(\frac{3}{5}\right) + 0\left(\frac{4}{5}\right) = -\frac{3}{5}$$

(b) (5 points) The equation for the tangent plane to the surface $\boldsymbol{S}$ at $\boldsymbol{P(0, 1, 3)}$ is

$\underline{x + z = 3}$.

Solution: The normal direction of the tangent plane to the surface S **at** $P(0, 1, 3)$ **is** $\vec{n} = \langle -f_x(0, 1), -f_y(0, 1), 1 \rangle = \langle -(-1), -(0), 1 \rangle = \langle 1, 0, 1 \rangle$. **So the equation of the tangent plane is**

$$(1)(x - 0) + (0)(y - 1) + (1)(z - 3) = 0 \iff x + z = 3$$

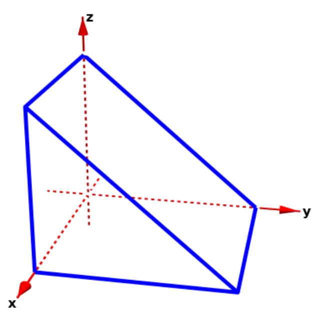
4. Consider the integral $\displaystyle\int_0^1 \int_0^2 \boldsymbol{y \, dy \, dx}$

(a) (4 points) Evaluate the integral. **Note: No partial credit for this problem.**

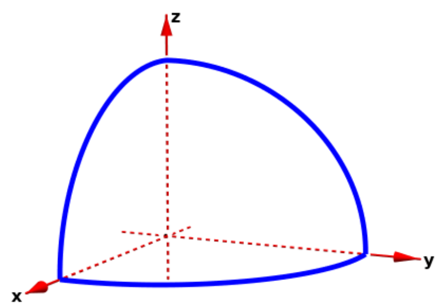
Solution: $\displaystyle\int_0^1 \int_0^2 y \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 \mathrm{d}x \int_0^2 y \, \mathrm{d}y = (1) \cdot \frac{y^2}{2}\bigg|_0^2 = \boxed{2}$

(b) (4 points) The integral above can be interpreted as the volume of a solid. Which of the following is the best representation of the solid? **The answer is (2).**

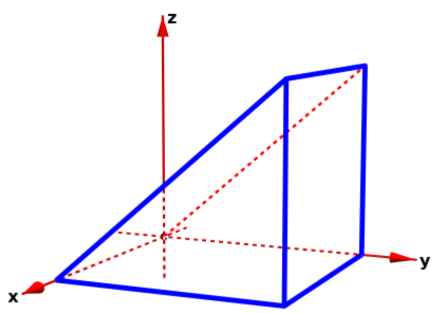
(1)



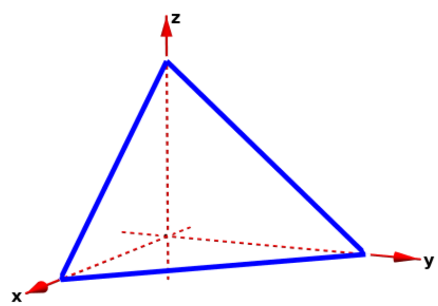
(4)



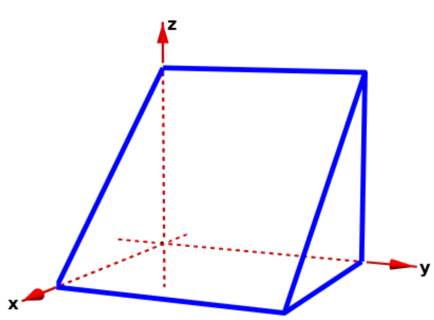
(2)



(5)



(3)

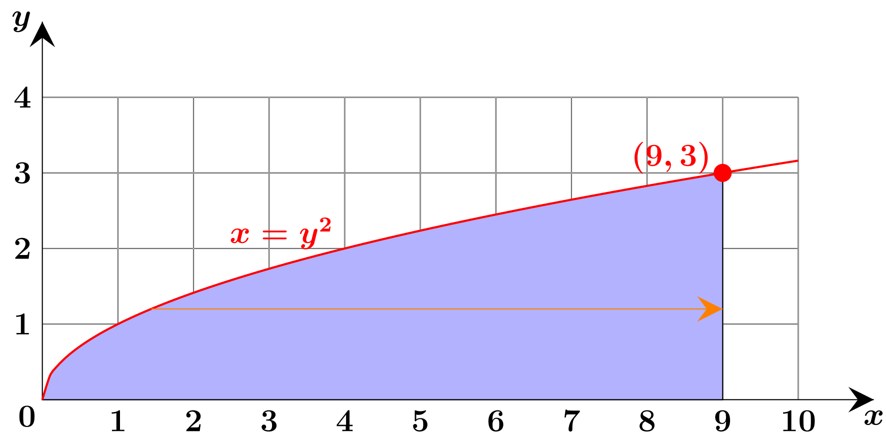


5. Consider the integral $\int_0^3 \int_{y^2}^9 \boldsymbol{y \sin(x^2)\, dx\, dy}$.

(a) (4 points) Sketch the region of integration.

Solution: **Based on the integral $\int_0^3 \int_{y^2}^9 y\sin(x^2)\, dx\, dy$, we can get the region D as follows,**

$$D = \{(x,y) | 0 \le y \le 3,\ y^2 \le x \le 9\}$$



(b) (8 points) Evaluate the integral.

Solution: The integration region can be expressed as type I and type II regions as follows,

$$D = \{(x,y) | 0 \le y \le 3,\ y^2 \le x \le 9\} = \{(x,y) | 0 \le x \le 9,\ 0 \le y \le \sqrt{x}\}$$

To evalute the integral, we need to exchange the integration order:

$$\int_0^3 \int_{y^2}^9 y\sin(x^2)\, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} y\sin(x^2)\, dy\, dx = \int_0^9 \sin(x^2)\, \frac{y^2}{2}\Big|_{x=0}^{y=\sqrt{x}} dx$$

$$= \int_0^9 \frac{1}{2} x\, \sin(x^2)\, dx \quad \left(\text{Let } u = x^2,\ du = 2x\, dx \quad \begin{array}{|c|c|c|} \hline x & 0 & 9 \\ \hline u & 0 & 81 \\ \hline \end{array} \right)$$

$$= \int_0^{81} \frac{1}{4} \sin u\, du = -\frac{1}{4} \cos u \Big|_0^{81} = \boxed{\frac{1}{4}[1 - \cos(81)]}$$

6. (12 points) Consider the function

$$f(x,y) = \begin{cases} \dfrac{y^2 + x}{y^2 + y\sqrt{x} + x} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Is f continuous at $(x,y) = (0,0)$? Justify your answer.

Solution: $f(0,0) = 1$. f is continuous at $(0,0) \iff \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Take a path $y = \sqrt{x}$ $(x \geq 0)$

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=\sqrt{x}}} f(x,y) = \lim_{x\to 0^+} \frac{(\sqrt{x})^2 + x}{(\sqrt{x})^2 + (\sqrt{x})^2 + x} = \lim_{x\to 0^+} \frac{2x}{3x} = \frac{2}{3} \neq f(0,0) = 1$$

So the function f is discontinuous.

Note 1: If we take another path, for example $x = 0$, then the limit is

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=0}} f(x,y) = \lim_{y\to 0} \frac{y^2}{y^2} = \lim_{y\to 0} 1 = 1 \neq \frac{2}{3}.$$

This means the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Note 2: There are infinitely many paths through $(0,0)$. If we take a path $y = kx$, where k is a constant, we can check that the limit is always 1 as k varies. This does not mean that the limit of f is 1 as $(x,y) \to (0,0)$.

Note 3: We may also take a path $x = y^2$, which passes through $(0,0)$.

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=y^2}} f(x,y) = \lim_{y\to 0} \frac{y^2 + y^2}{y^2 + y|y| + y^2} = \lim_{y\to 0} \frac{2y}{2y + |y|}.$$

This limit doesn't exist since

$$\lim_{y\to 0^+} \frac{2y}{2y + |y|} = \lim_{y\to 0^+} \frac{2y}{2y + y} = \frac{2}{3}, \quad \lim_{y\to 0^-} \frac{2y}{2y + |y|} = \lim_{y\to 0^-} \frac{2y}{2y - y} = 2.$$

This also shows that the limit of f does not exist as $(x,y) \to (0,0)$.

Hence f is not continuous at $(0,0)$.

7. (12 points) Find all critical points of the function

$$\boldsymbol{f(x,y) = 2x^3 - 6xy + y^2 + 7}$$

and classify each as a local maximum, local minimum, saddle point, or not enough information. Justify your answer.

Solution: $f_x = \dfrac{\partial}{\partial x}\left(2x^3 - 6xy + y^2 + 7\right) = 6x^2 - 6y$

$f_y = \dfrac{\partial}{\partial y}\left(2x^3 - 6xy + y^2 + 7\right) = 2y - 6x$

Solving $\nabla f = \langle 0, 0\rangle \iff \begin{cases} 6x^2 - 6y = 0 & ① \\ 2y - 6x = 0 & ② \end{cases}$ **for** (x, y).

From ② we get $y = 3x$. **Plug it into ① and we get** $6x^2 - 18x = 0$, $6x(x-3) = 0$, $x = 0$ **or** $x = 3$. **Plug them into** $y = 3x$ **and we obtain two critical points:** $(0,0)$ **and** $(3,9)$.

To classify them, we use the second derivative test.

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(6x^2 - 6y\right) = 6$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(2y - 6x\right) = 2$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(6x^2 - 6y\right) = -6$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 = (12x)(2) - (-6)^2 = 24x - 36$$

- **At** $(0,0)$, $D(0,0) = 24(0) - 36 = -36 < 0$, **and so** $(0,0)$ **is a saddle point.**
- **At** $(3,9)$, $D(3,9) = 24(3) - 36 = 72 - 36 = 36 > 0$, $f_{xx}(3,9) = 6 > 0$, **so** $(3,9)$ **is a local minimum point.**

8. You are hiking up a mountain whose elevation is given roughly by

$$f(x,y) = x^2 - y^3 + xy^2 + 2.$$

You are at $(1,-1,5)$ in this model.

(a) (6 points) What is the steepest uphill direction from your position?

Solution: The max rate of change of f occurs along the direction of the gradient. For this problem, the steepest uphill direction from your position $(1,-1,5)$ is $\nabla f(1,-1)$.

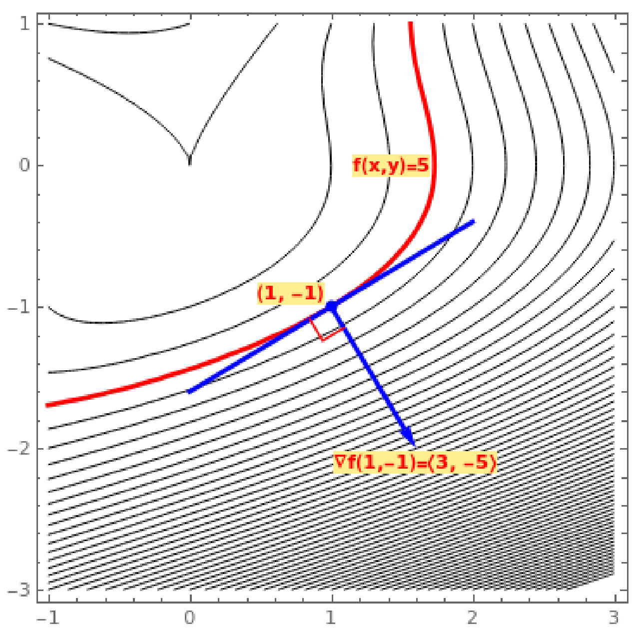
$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(x^2 - y^3 + xy^2 + 2\right) = 2x + y^2 \\ f_y &= \frac{\partial}{\partial y}\left(x^2 - y^3 + xy^2 + 2\right) = -3y^2 + 2xy \\ \nabla f &= \langle f_x, f_y\rangle = \left\langle 2x + y^2, -3y^2 + 2xy\right\rangle \\ \nabla f(1,-1) &= \langle 3, -5\rangle \end{aligned}$$

(b) (4 points) Suppose you wish to move from your current location while maintaining your current elevation. Find a vector parallel to the direction in which you should move.

Solution: The gradient direction is orthogonal or perpendicular to the level curve or the contour line. So in order to maintain your current elevation from your current location, we find the direction $\vec{v} = \langle v_1, v_2\rangle$ such that $\vec{v} \perp \nabla f(1,-1)$.

$$\begin{aligned} \vec{v} &\perp \nabla f(1,-1) \\ \vec{v} \cdot \nabla f(1,-1) &= 0 \\ \langle v_1, v_2\rangle \cdot \langle 3, -5\rangle &= 0 \\ 3v_1 - 5v_2 &= 0 \\ \implies v_1 &= \frac{5}{3}v_2 \\ \vec{v} = \left\langle \frac{5}{3}v_2, v_2\right\rangle &= \frac{v_2}{3}\langle 5, 3\rangle \end{aligned}$$

where v_2 can be any nonzero number.



9. (10 points) Use the **method of Lagrange multipliers** to find the extreme values of the function $f(x, y) = 2x - y^2$ subject to the constraint $x^2 + 3y^2 = 36$.

Solution: Let $g(x, y) = x^2 + 3y^2$. Then we solve

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases} \iff \begin{cases} \langle 2,\ -2y\rangle = \lambda\,\langle 2x,\ 6y\rangle \\ x^2 + 3y^2 = 36 \end{cases} \iff \begin{cases} 2 = 2x\lambda \text{ ①} \\ -2y = 6y\lambda \text{ ②} \\ x^2 + 3y^2 = 36 \quad \text{③} \end{cases}$$

From ②, $-2y - 6y\lambda = 0 \iff -2y(1 + 3\lambda) = 0 \iff y = 0$ or $\lambda = -\frac{1}{3}$

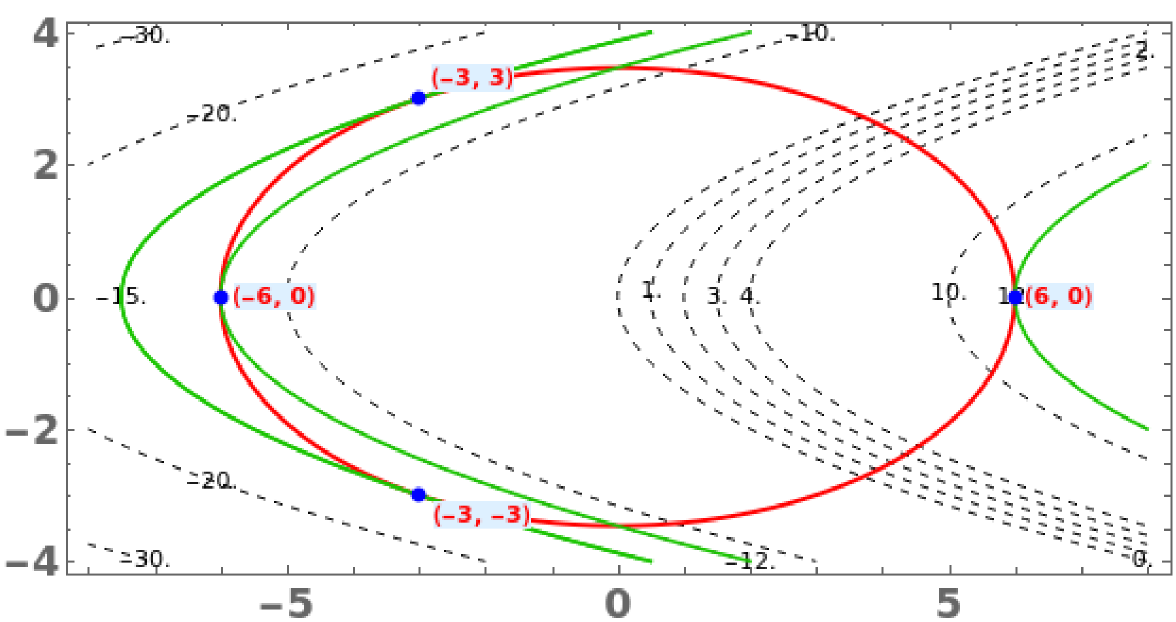
(a) $y = 0$: $\xrightarrow{③} x^2 + 0^2 = 36 \implies x = \pm 6 \xrightarrow{①} \lambda = \frac{1}{6}$ if $x = 6$, $\lambda = -\frac{1}{6}$ if $x = -6$.
So we get two points on the boundary: $(x, y, \lambda) = (6, 0, \frac{1}{6})$, $(-6, 0, -\frac{1}{6})$.

(b) $\lambda = -\frac{1}{3}$: $\xrightarrow{①} x = -3 \xrightarrow{③} (-3)^2 + y^2 = 36 \implies y^2 = 9 \implies y = \pm 3$
So we get two points subject to the constraint in this case:
$(x, y) = (-3, 3)\,,\ (-3, -3)$.

Evaluate the function values of f at the points obtained in the above steps.

$$f(6,0) = 2(6) - (0)^2 = 12, \qquad f(-6,0) = 2(-6) - (0)^2 = -12$$
$$f(-3,3) = 2(-3) - (3)^2 = -15,\ f(-3,-3) = 2(-3) - (-3)^2 = -15$$

So after comparing these values, we get
f attains its max value 12 at $(6, 0)$, and its min value -15 at $(-3, \pm 3)$.



10. Suppose that $z = F(x, y)$ and that $x = X(s, t)$ and $y = Y(s, t)$, where F, X and Y all have continuous partial derivatives at all points.

Suppose that the following facts are given:

$\dfrac{\partial z}{\partial x}(1, 11) = b$	$\dfrac{\partial z}{\partial x}(c, d) = 5$	$\dfrac{\partial z}{\partial y}(1, 11) = 11$	$\dfrac{\partial z}{\partial y}(c, d) = 12$
$\dfrac{\partial x}{\partial s}(c, d) = 10$	$\dfrac{\partial x}{\partial t}(c, d) = 7$	$\dfrac{\partial y}{\partial s}(c, d) = a$	$\dfrac{\partial z}{\partial t}(c, d) = 0$
$X(a, b) = -1$	$Y(a, b) = -7$	$X(c, d) = 1$	$Y(c, d) = 11$

Note: One can view z as a function of x and y, and one can view z as a function of s and t.

(a) (5 points) Find $\dfrac{\partial z}{\partial s}(c, d)$

Solution: $X(c, d) = 1$, $Y(c, d) = 11$. By Chain Rule,

$$\begin{aligned}
\frac{\partial z}{\partial s}(c, d) &= F_x(X(c, d), Y(c, d)) \cdot \frac{\partial X}{\partial s}(c, d) + F_y(X(c, d), Y(c, d)) \cdot \frac{\partial Y}{\partial s}(c, d) \\
&= \frac{\partial z}{\partial x}(1, 11) \cdot \frac{\partial x}{\partial s}(c, d) + \frac{\partial z}{\partial y}(1, 11) \cdot \frac{\partial y}{\partial s}(c, d) \\
&= (b)(10) + (11)(a) = 11a + 10b
\end{aligned}$$

(b) (5 points) Find $\dfrac{\partial y}{\partial t}(c, d)$

Solution: Note $X(c, d) = 1$, $Y(c, d) = 11$. Differentiate both sides of $z = F(X(s, t), F(s, t))$ with respect to t, and we get

$$\frac{\partial z}{\partial s} = F_x(X(s, t), F(s, t)) \cdot \frac{\partial X}{\partial t} + F_y(X(s, t), F(s, t)) \cdot \frac{\partial Y}{\partial t}$$

$$\frac{\partial z}{\partial s}(c, d) = F_x(X(c, d), F(c, d)) \cdot \frac{\partial x}{\partial t}(c, d) F_y(X(c, d), F(c, d)) \cdot \frac{\partial y}{\partial t}(c, d)$$

$$\frac{\partial z}{\partial s}(c, d) = \frac{\partial z}{\partial x}(1, 11) \cdot \frac{\partial x}{\partial t}(c, d) + \frac{\partial z}{\partial y}(1, 11) \cdot \frac{\partial y}{\partial t}(c, d)$$

So we have

$$0 = (b)(7) + (11)\frac{\partial y}{\partial t}(c, d) \Longrightarrow \frac{\partial y}{\partial t}(c, d) = -\frac{7b}{11}$$