Math 2400, Midterm 1

September 24, 2018

PRINT YOUR NAME: ______________________________

PRINT INSTRUCTOR'S NAME: ______________________________

Mark your section/instructor:

☐	Section 001	Kevin Berg	8:00–8:50
☐	Section 002	Philip Kopel	8:00–8:50
☐	Section 003	Daniel Martin	8:00–8:50
☐	Section 004	Albert Bronstein	9:00–9:50
☐	Section 005	Mark Pullins	9:00–9:50
☐	Section 006	Xingzhou Yang	9:00–9:50
☐	Section 007	Martin Walter	10:00–10:50
☐	Section 008	Kevin Manley	10:00–10:50
☐	Section 009	Albert Bronstein	1:00–1:50
☐	Section 010	Martin Walter	1:00–1:50
☐	Section 011	Xingzhou Yang	2:00–2:50
☐	Section 012	Taylor Klotz	2:00–2:50
☐	Section 013	Xingzhou Yang	3:00–3:50
☐	Section 014	Braden Balentine	4:00–4:50
☐	Section 015	Caroline Matson	4:00–4:50

Question	Points	Score
1	10	
2	12	
3	12	
4	4	
5	8	
6	6	
7	8	
8	10	
9	10	
10	10	
11	10	
Total:	100	

Honor Code

On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work.

- No calculators or cell phones or other electronic devices allowed at any time.
- Show all your reasoning and work for full credit, except where otherwise indicated. Use full mathematical or English sentences.
- You have 90 minutes and the exam is 100 points.
- You do not need to simplify numerical expressions. For example leave fractions like $\mathbf{100/7}$ or expressions like $\mathbf{\ln(3)/2}$ as is.
- When done, give your exam to your instructor, who will mark your name off on a photo roster.
- We hope you show us your best work!

1. (10 points) **Note: No partial credit for this problem.**
Let $\vec{a} = \langle 1, 2, 1\rangle$, $\vec{b} = \langle -2, 1, -3\rangle$. Compute

(a) $|\vec{a}| = \underline{\sqrt{6}}$.

Solution: $|\vec{a}| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$.

(b) $3\vec{a} - \vec{b} = \underline{\langle 5,\ 5,\ 6\rangle}$.

Solution: $3\vec{a} - \vec{b} = 3\langle 1, 2, 1\rangle - \langle -2, 1, -3\rangle = \langle 3+2,\ 6-1,\ 3+3\rangle$
$= \langle 5,\ 5,\ 6\rangle$.

(c) $\vec{a} \cdot \vec{b} = \underline{-3}$.

Solution: $\vec{a} \cdot \vec{b} = \langle 1, 2, 1\rangle \cdot \langle -2, 1, -3\rangle$
$= (1)(-2) + (2)(1) + (1)(-3) = -3.$

(d) $\vec{a} \times \vec{b} = \underline{\langle -7,\ 1,\ 5\rangle}$.

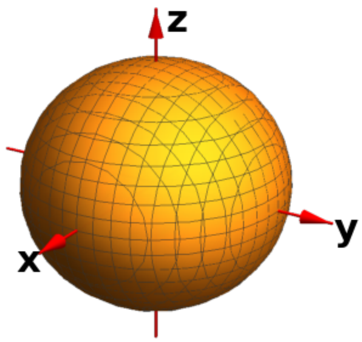
Solution: $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ -2 & 1 & -3 \end{vmatrix}$
$= \vec{i}\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}$
$= (-6-1)\vec{i} - (-3+2)\vec{j} + (1+4)\vec{k}$
$= -7\vec{i} + \vec{j} + 5\vec{k} = \langle -7,\ 1,\ 5\rangle$

(e) $\text{proj}_{\vec{a}}\vec{b} = \underline{-\frac{1}{2}\langle 1,\ 2,\ 1\rangle = \left\langle -\frac{1}{2},\ -1,\ -\frac{1}{2}\right\rangle}$.

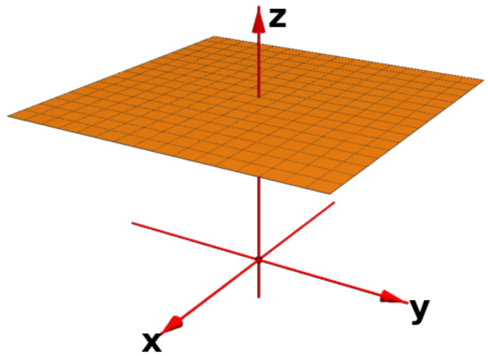
Solution: $\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a} = \dfrac{-3}{6}\langle 1, 2, 1\rangle = -\dfrac{1}{2}\langle 1,\ 2,\ 1\rangle$
$= \left\langle -\frac{1}{2},\ -1,\ -\frac{1}{2}\right\rangle$

2. (12 points) Match each 3D surface with one of the equations. Not all equations will be matched.

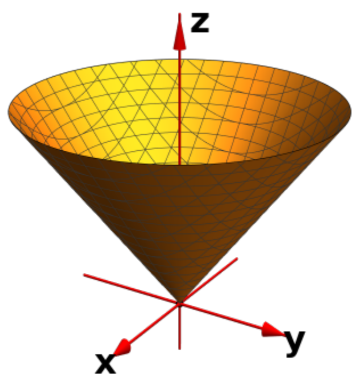
(a) **(1)**



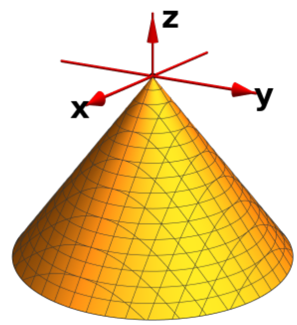
(b) **(6)**



(c) **(5)**



(d) **(3)**



Let $(\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi})$ be spherical coordinates.

(1) $\boldsymbol{\rho = 3}$
Cartesian Eqn:
$\boldsymbol{x^2 + y^2 + z^2 = 9}$

(2) $\boldsymbol{\theta = \frac{\pi}{3}}$
Cartesian Eqn:
$\boldsymbol{y = \sqrt{3}x}$.

(3) $\boldsymbol{\phi = \frac{2\pi}{3}}$
Cartesian Eqn:
$\boldsymbol{x^2 + y^2 = 3z^2}$,
$\boldsymbol{z \leq 0}$

(4) $\boldsymbol{\rho = \sec\theta}$
Cartesian Eqn:
$\boldsymbol{\frac{x^2 + y^2}{x^2 + y^2 + z^2} = x^2}$

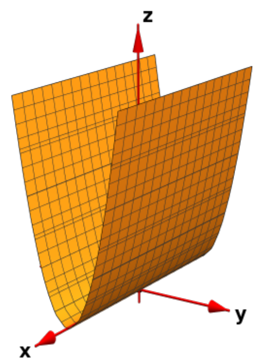
(5) $\boldsymbol{\phi = \frac{\pi}{3}}$
Cartesian Eqn:
$\boldsymbol{x^2 + y^2 = 3z^2}$,
$\boldsymbol{z \geq 0}$

(6) $\boldsymbol{\rho = \sec\phi}$
Cartesian Eqn:
$\boldsymbol{z = 1}$
$(\boldsymbol{\rho\cos\phi = 1})$

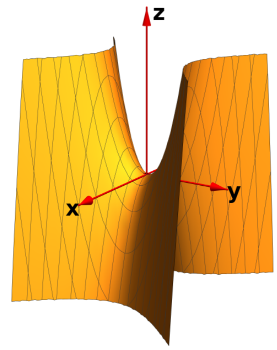
Note: In spherical coord., $\boldsymbol{x = \rho\sin\phi\cos\theta}$, $\boldsymbol{y = \rho\sin\phi\sin\theta}$, $\boldsymbol{z = \rho\cos\phi}$.
$\boldsymbol{x^2+y^2+z^2 = \rho^2}$, $\boldsymbol{\tan\theta = \frac{y}{x}}$, $\boldsymbol{\sin^2\phi = \frac{x^2 + y^2}{x^2 + y^2 + z^2}}$, $\boldsymbol{\cos^2\phi = \frac{z^2}{x^2 + y^2 + z^2}}$.

3. (12 points) Match each 3D surface with one of the equations on the right side. Not all equations will be matched.

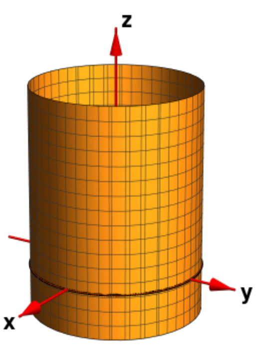
(a) **(8)**



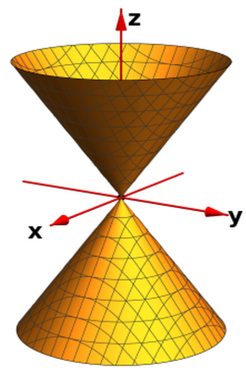
(b) **(5)**



(c) **(3)**



(d) **(1)**



(1) $\boldsymbol{x^2 + y^2 - z^2 = 0}$

(2) $\boldsymbol{x^2 - y^2 + z^2 = 0}$

(3) $\boldsymbol{x^2 + y^2 - 4 = 0}$

(4) $\boldsymbol{x^2 - y^2 - 4 = 0}$

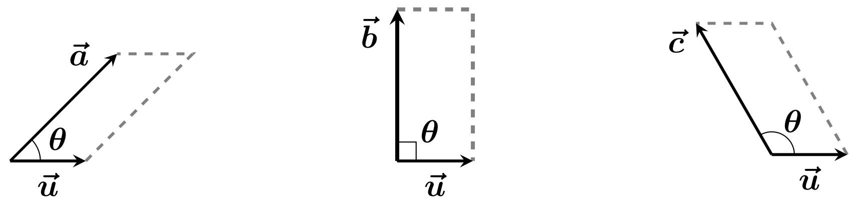
(5) $\boldsymbol{z = y^2 - x^2}$

(6) $\boldsymbol{z = x^2 - y^2}$

(7) $\boldsymbol{z = x^2}$

(8) $\boldsymbol{z = y^2}$

4. (4 points) In the pictures below assume that $|\vec{\boldsymbol{a}}| = |\vec{\boldsymbol{b}}| = |\vec{\boldsymbol{c}}|$ and that $\vec{\boldsymbol{u}}$ is a unit vector. Use the parallelograms below to rank $|\vec{\boldsymbol{a}} \times \vec{\boldsymbol{u}}|$, $|\vec{\boldsymbol{b}} \times \vec{\boldsymbol{u}}|$, and $|\vec{\boldsymbol{c}} \times \vec{\boldsymbol{u}}|$ from smallest to largest.



$\theta = \frac{\pi}{4}$ $\qquad$ $\theta = \frac{\pi}{2}$ $\qquad$ $\theta = \frac{2\pi}{3}$

Circle one of the following.

(a) $|\vec{a} \times \vec{u}| \leq |\vec{b} \times \vec{u}| \leq |\vec{c} \times \vec{u}|$ **wrong answer**

(b) $|\vec{\boldsymbol{a}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{c}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{b}} \times \vec{\boldsymbol{u}}|$ **Correct Answer: (b)**

(c) $|\vec{\boldsymbol{b}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{a}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{c}} \times \vec{\boldsymbol{u}}|$

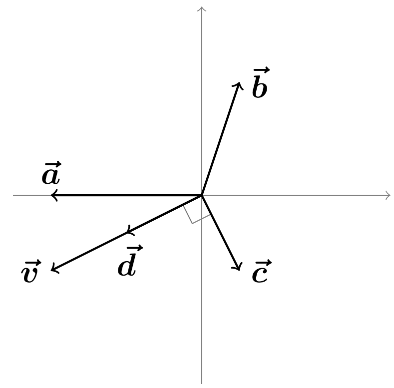
(d) $|\vec{\boldsymbol{b}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{c}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{a}} \times \vec{\boldsymbol{u}}|$

(e) $|\vec{\boldsymbol{c}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{a}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{b}} \times \vec{\boldsymbol{u}}|$

(f) $|\vec{\boldsymbol{c}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{b}} \times \vec{\boldsymbol{u}}| \leq |\vec{\boldsymbol{a}} \times \vec{\boldsymbol{u}}|$

Note: The area of the parallelogram spanned by vectors $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin\theta$, where θ is the angle between the two vectors.

5. (8 points) Consider the following vectors $\vec{v}$, $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ in the xy-plane.



Circle whether each of the following is true or false.

(a) $\boxed{\textbf{TRUE}}$ or **FALSE**: $\vec{v} \cdot \vec{a} \geq \vec{v} \cdot \vec{d}$

(b) $\boxed{\textbf{TRUE}}$ or **FALSE**: $\vec{v} \cdot \vec{c} = 0$

(c) **TRUE** or $\boxed{\textbf{FALSE}}$: $\vec{v} \cdot \vec{a} = \vec{v} \cdot \vec{b}$

(d) **TRUE** or $\boxed{\textbf{FALSE}}$: $\vec{v} \cdot \vec{d} = |\vec{v}|^2$

6. (6 points) Let $\boldsymbol{C}$: $\vec{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ be any space curve such that $|\vec{r}\,'(t)| = 1$. If $3 \le t \le 5$ then what is the length of $\boldsymbol{C}$ from $\vec{r}(3)$ to $\vec{r}(5)$?

Solution:

$$L = \int_3^5 |\vec{r}\,'(t)|\, dt = \int_3^5 1\, dt = (5-3) = \boxed{2}$$

7. (8 points) Find a parametric representation of the part of the plane $z = x + 2$ that lies inside the cylinder $x^2 + y^2 = 1$.

Solution:

$$\begin{cases} x &= x \\ y &= y \\ z &= x+2 \end{cases}, \quad x^2 + y^2 \le 1$$

$$\text{or} \quad \begin{cases} x &= u \\ y &= v \\ z &= u+2 \end{cases}, \quad u^2 + v^2 \le 1$$

Solution 2: In cylindrical coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Plug them into the plane equation, and we have

$$z = r\cos\theta + 2$$

Now we get the paramatric equations for the surface as follows,

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= r\cos\theta + 2 \end{cases}, \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi$$

8. (10 points) Consider the space curve $\boldsymbol{C}$: $\vec{r}(t) = \langle t \ln t,\ 2t,\ t^2 \rangle$, where $t > 0$.

(a) Calculate $\dfrac{d}{dt}[\vec{r}(t)]$.

Solution:

$$\frac{d}{dt}[\vec{r}(t)] = \frac{d}{dt}\langle t\ln t,\ 2t,\ t^2\rangle = \left\langle \frac{d}{dt}(t\ln t),\ \frac{d}{dt}(2t),\ \frac{d}{dt}(t^2) \right\rangle$$
$$= \boxed{\langle 1 + \ln t,\ 2,\ 2t\rangle}$$

(b) Find the parametric equations of the line tangent to $\boldsymbol{C}$ at the point $(0, 2, 1)$.

Solution: To find the corresponding t to the point, we solve $\langle t\ln t,\ 2t,\ t^2\rangle = \langle 0, 2, 1\rangle$, or $t\ln t = 0$, $2t = 2$, $t^2 = 1$, for t. So we get only one root $t = 1$. The tangential direction of the tangent line is $\vec{r}\,'(1) = \langle 1 + \ln 1, 2, 2(1)\rangle = \langle 1, 2, 2\rangle$. So the parametric equations of the line tangent to $\boldsymbol{C}$ at the point $(0, 2, 1)$ is

$$\boxed{x = t,\ y = 2 + 2t,\ z = 1 + 2t.}$$

9. (10 points) The positions of two particles at time t, for $t \geq 0$, are given by $\vec{r}_1(t) = \langle t,\ t^2 + 2t,\ 1 \rangle$ and $\vec{r}_2(t) = \langle 2t - 1,\ 4t^2 - 1,\ \cos(\pi t) \rangle$, respectively. Do the two particles collide? If so, then at what time(s) do they collide? If not, justify why not.

Solution: If the two particles collide, they should have the same position at time t, in other words, $\vec{r}_1(t) = \vec{r}_2(t)$. So

$$\langle t, t^2 + 2t, 1 \rangle = \langle 2t - 1, 4t^2 - 1, \cos(\pi t) \rangle$$

$$\begin{cases} t = 2t - 1 & ① \\ t^2 + 2t = 4t^2 - 1 & ② \\ 1 = \cos(\pi t) & ③ \end{cases}$$

From ①, we get $t = 1$, but it does not satisfy ③ since $\cos(\pi) = -1 \neq 1$. So the two particles **never collide**.

10. (10 points) Find an equation for the plane through the point $\boldsymbol{P(2,3,0)}$ and parallel to a plane determined by vectors $\vec{\boldsymbol{u}} = \langle \mathbf{1},\ \mathbf{2},\ \mathbf{3} \rangle$ and $\vec{\boldsymbol{v}} = \langle \mathbf{3},\ \mathbf{4},\ \mathbf{5} \rangle$.

Solution: The normal direction of the plane determined by $\vec{\boldsymbol{u}}$ and $\boldsymbol{v}$ is

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= \vec{i}(10 - 12) - \vec{j}(5 - 9) + \vec{k}(4 - 6)$$

$$= -2\vec{i} + 4\vec{j} - 2\vec{k} = \langle -2,\ 4,\ -2 \rangle$$

So the equation of the plane through $\boldsymbol{P(2,3,0)}$ and parallel to the plane determined by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ is

$$\boxed{-2(x-2) + 4(y-3) - 2(z-0) = 0}, \quad \text{or} \quad \boxed{x - 2y + z + 4 = 0}$$

11. (10 points) Given $\boldsymbol{P(1,\ 2,\ 3)}$ and a line $\boldsymbol{L}:\ \dfrac{\boldsymbol{x-1}}{\boldsymbol{2}} = \dfrac{\boldsymbol{y+2}}{\boldsymbol{1}} = \dfrac{\boldsymbol{z-3}}{\boldsymbol{2}}$, find the shortest distance between the point $\boldsymbol{P}$ and the line $\boldsymbol{L}$.

Solution: Let $\vec{\boldsymbol{u}}$ be the direction vector of $\boldsymbol{L}$. Then $\vec{\boldsymbol{u}} = \langle \mathbf{2, 1, 2} \rangle \parallel \boldsymbol{L}$. Note that the point $\boldsymbol{Q}(\mathbf{1, -2, 3})$ is on $\boldsymbol{L}$, and

$$\overrightarrow{\boldsymbol{PQ}} = \overrightarrow{\boldsymbol{OQ}} - \overrightarrow{\boldsymbol{OP}} = \langle \mathbf{1, -2, 3} \rangle - \langle \mathbf{1, 2, 3} \rangle = \langle \mathbf{0, -4, 0} \rangle \, .$$

The projection of $\overrightarrow{\boldsymbol{PQ}}$ onto $\boldsymbol{L}$ or $\vec{\boldsymbol{u}}$ is

$$\begin{aligned} \text{Proj}_{\vec{u}} \overrightarrow{\boldsymbol{PQ}} &= \frac{\vec{\boldsymbol{u}} \cdot \overrightarrow{\boldsymbol{PQ}}}{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}} \vec{\boldsymbol{u}} = \frac{\mathbf{(2)(0) + (1)(-4) + (2)(0)}}{\mathbf{2^2 + 1^2 + 2^2}} \vec{\boldsymbol{u}} \\ &= -\frac{\mathbf{4}}{\mathbf{9}} \vec{\boldsymbol{u}} = -\frac{\mathbf{4}}{\mathbf{9}} \langle \mathbf{2, 1, 2} \rangle \end{aligned}$$

So the orthogonal projection of $\overrightarrow{\boldsymbol{PQ}}$ onto $\boldsymbol{L}$ is

$$\begin{aligned} \text{Orth}_{\vec{u}} \overrightarrow{\boldsymbol{PQ}} &= \overrightarrow{\boldsymbol{PQ}} - \text{Proj}_{\vec{u}} \overrightarrow{\boldsymbol{PQ}} = \langle \mathbf{0, -4, 0} \rangle - \left(-\frac{\mathbf{4}}{\mathbf{9}} \right) \langle \mathbf{2, 1, 2} \rangle \\ &= \left\langle \mathbf{0} + \frac{\mathbf{8}}{\mathbf{9}}, \mathbf{-4} + \frac{\mathbf{4}}{\mathbf{9}}, \mathbf{0} + \frac{\mathbf{8}}{\mathbf{9}} \right\rangle = \left\langle \frac{\mathbf{8}}{\mathbf{9}}, -\frac{\mathbf{32}}{\mathbf{9}}, \frac{\mathbf{8}}{\mathbf{9}} \right\rangle \\ &= \frac{\mathbf{8}}{\mathbf{9}} \langle \mathbf{1, -4, 1} \rangle \end{aligned}$$

The distance between the point $\boldsymbol{P}$ and the line $\boldsymbol{L}$ is the magnitude of the orthogonal vector, i.e.,

$$\text{dist}(\boldsymbol{P}, \boldsymbol{L}) = \left| \text{Orth}_{\vec{u}} \overrightarrow{\boldsymbol{PQ}} \right| = \frac{\mathbf{8}}{\mathbf{9}} \sqrt{\mathbf{1^2 + (-4)^2 + 1^2}} = \frac{\mathbf{8}}{\mathbf{9}} \sqrt{\mathbf{18}} = \boxed{\frac{\mathbf{8\sqrt{2}}}{\mathbf{3}}}$$