UNIVERSITY OF COLORADO, BOULDER
Department of Mathematics
MATH 2400, Calculus 3
Final Exam (100 points), 12/15/16

YOUR NAME:

YOUR STUDENT ID:

CIRCLE SECTION NUMBER AND THE INSTRUCTOR'S NAME [2pts]:

1-Steindl
2-Watts
3-Frinak
4-Moorhead
5-Bronstein
6-Weinell
7-Englander
8-Healy
9-Rosenbaum
10-Steindl
11-Huang
12-Huang

IMPORTANT:

1. Please write all your answers on this exam. **Show all work clearly and in order, and box your answers.** Please justify your answers. A final or numerical answer without explanation is not sufficient for full credit.

2. The exam is closed book/notes. Tablets, cell phones, and other electronic devices are not permitted. However, simple calculators are OK.

3. Throughout this exam, please provide exact answers where possible. That is: if the answer is 1/2, do not write 0.499 or something of that sort; if the answer is π, do not write 3.14159 or something of that sort.

4. Make sure you write an arrow on top of vector quantities (or underline them) to differentiate them from scalar quantities (numbers).

5. No explanations are needed in the multiple choice part, but full explanation is needed in the rest of the questions.

Questions start on the next page. GOOD LUCK!

1 Multiple choice part:

In the next 10 questions, **circle the correct answer**, without explanation.

1. **(5 pts)** Evaluate the integral $\int_{\mathcal{C}} (x+y)\, ds$, where $\mathcal{C}$ is the half of the circle $x^2 + y^2 = 2$ in the upper half plane. The integral equals

   (a) 4
   (b) 5
   (c) 6
   (d) 7
   (e) 8

2. **(5 pts)** Assume that a cube has unit side length and lies in the first octant with faces parallel to the coordinate planes and one vertex at the origin. Then the vector $\vec{u} = \langle 1, 0, 0 \rangle$ describes an edge of the cube, and the vector $\vec{v} = \langle 1, 1, 1 \rangle$ describes the skew diagonal. The cosine of the angle between $\vec{u}$ and $\vec{v}$ is

   (a) $\frac{2}{\sqrt{6}}$
   (b) $\frac{2}{6}$
   (c) $\frac{1}{\sqrt{6}}$
   (d) $\frac{1}{6}$

3. **(5 pts)** Find the volume of the rectangular parallelepiped determined (spanned) by the vectors $\langle 1, 2, 1 \rangle$, $\langle 1, 3, 2 \rangle$ and $\langle 3, 1, 1 \rangle$. The volume is

   (a) 2
   (b) 3
   (c) 0
   (d) 1

4. **(5 pts)** The equation of the plane containing the points $(0, 1, 1)$, $(-1, 2, 1)$ and $(1, 1, 0)$ is

   (a) $x + 2(y-1) + (z-2) = 0$
   (b) $(x-1) - 2(y-2) - (z+1) = 0$
   (c) $(x+1) + (y-2) + (z-1) = 0$
   (d) $(y-1) + 2(z-1) = 0$

5. **(5 pts)** Consider the hyperboloid of one sheet given by the equation

$$x^2 + (2y)^2 - z^2 = 1.$$

Which of the following **best** describes the traces of this surface? Recall that a horizontal (vertical) trace is an intersection of this surface with the plane $z = c$ (x or $y = c$) for some constant c.

   (a) Horizontal traces are circles, vertical traces are hyperbolas.
   (b) Horizontal traces are (non-circular) ellipses, vertical traces are hyperbolas.
   (c) Horizontal traces are pairs of lines, vertical traces are circles.
   (d) Horizontal traces are hyperbolas, vertical traces are ellipses.

6. **(5 pts)** Let P be the point in $\mathbb{R}^3$ with cylindrical coordinates $(r, \theta, z) = (\sqrt{2}, \pi/4, \sqrt{2})$. What are the spherical coordinates for P? They are

   (a) $(\rho, \theta, \phi) = (\sqrt{8}, \pi/4, \pi/4)$
   (b) $(\rho, \theta, \phi) = (2, \pi/4, \pi/4)$
   (c) $(\rho, \theta, \phi) = (2, \pi/4, \pi/2)$
   (d) $(\rho, \theta, \phi) = (\sqrt{8}, \pi/4, \pi/2)$

7. **(5 pts)** Let $g(x, y, z)$ be a differentiable function with $\nabla g = \langle x, x, \sin(z^2)\rangle$. Assume that

$$x = 2s + t, \quad y = st, \quad z = t^2, \text{ for } -5 \leq s, t \leq 5$$

It follows that $\dfrac{\partial g}{\partial s}$ at $(s, t) = (-1, 2)$ is

   (a) -4
   (b) -2
   (c) 0
   (d) 2

8. **(5 pts)** Let $f(x, y) = x^2 + xy^2$. Then the directional derivative $D_{\vec{j}} f(1, 1)$, where $\vec{j} = \langle 0, 1\rangle$, is

   (a) 0
   (b) 1
   (c) 2
   (d) 3

9. **(5 pts)** Let h be a function on $\mathbb{R}^2$, such that the second partial derivatives exist and are continuous, with a critical point at $(4,1)$. Assume that $h_{xx}(4,1) = -2$, $h_{yy}(4,1) = -5$, and $h_{xy}(4,1) = 3$. Which of the following is true?

   (a) h has a local maximum at $(4,1)$.

   (b) h has a local minimum at $(4,1)$.

   (c) h has a saddle point at $(4,1)$.

   (d) None of the above are necessarily true.

10. **(5 pts)** Consider the transformation T from the uv-plane to the xy-plane, given by
$$x = 2u - 5v, \quad y = u - 3v.$$
Let R be a region in the uv-plane with area 5. The area of the image of R under the transformation T in the xy-plane equals

    (a) 5/7

    (b) 5

    (c) 7

    (d) 35

2 Regular questions part:

In the next 4 questions you should give full explanation.

11. **(12 pts)** Let $\mathcal{S}$ be the sphere $x^2 + y^2 + z^2 = 25$, oriented towards the origin, and let $\vec{F}(x, y, z) = \langle 3xy^2 + z, 3x^2y + z^2, x^2y^2 + z^3 \rangle$. Calculate the flux of $\vec{F}$ through $\mathcal{S}$.

12. **(12 pts)** Let $\mathcal{S}$ be a surface given by a part of a paraboloid as

$$\mathcal{S} := \{(x, y, z) \in \mathbb{R}^3 \mid y = x^2 + z^2;\ y < 1\}.$$

Let

$$\vec{F}(x, y, z) = \langle y e^{\cos x} + 3z, x e^{\sin z}, e^{\tan y} \rangle.$$

Find the flux of $\vec{G} := \nabla \times \vec{F} = \mathsf{curl}\, \vec{F}$ across $\mathcal{S}$ in the outward direction (pointing away from the inside of the paraboloid).

13. **(12 pts)**

(a) Show that the following vector field on $\mathbb{R}^2$ is conservative, by finding a potential function:

$$\vec{F}(x, y) = \frac{1}{(x^3 + y^3)^{4/3}} \langle x^2, y^2 \rangle.$$

(b) Evaluate $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$, where $\mathcal{C}$ is the curve

$$\vec{r}(t) = \left\langle e^{\sin(\pi t)}, \frac{2t}{1 + t^2} \right\rangle, \qquad t \in [0, 1].$$

14. **(12 pts)** Let

$$\vec{F}(x,y,z) = \left\langle \sin x - \frac{y^3}{3}, \cos y + \frac{x^3}{3}, xyz \right\rangle.$$

Using Stokes's Theorem, compute $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$, where $\mathcal{C}$ is the curve in which the cone $z^2 = x^2 + y^2$ intersects the plane $z = 1$. (Here $\mathcal{C}$ is oriented counterclockwise when viewed from above).