MATH 2400: Calculus 3, Fall 2014
Midterm 2

October 15, 2014

NAME AND SIGNATURE:

"On my honor, as a University of Colorado at Boulder student, I have neither given nor received unauthorized assistance on this work."

Circle your section.

001 G. Rea .... (9am)

002 J. Migler .... (10am)

003 I. Mishev .... (11am)

004 M. Roy .... (12pm)

005 T. Davison .... (1pm)

006 C. Farsi .... (2pm)

007 D. Monk .... (3pm)

008 K. Havasi .... (9am)

009 J. Nishikawa .... (11am)

You must show all of your work. Please write legibly and box your answers. The use of calculators, books, notes, etc. is not permitted on this exam. Please provide exact answers when possible. For example, if the answer is π, write the symbol "π" and not the decimal 3.14159. . . .

Question	Points	Score
1	15	
2	10	
3	20	
4	20	
5	20	
6	15	
Total:	100	

1. (15 points) Find the length of the curve $\underline{r}(t) = \langle 2t^{3/2}, \cos(2t), \sin(2t)\rangle, 0 \le t \le 1$.

   Solution
   First, we need the derivative:
   $$\underline{r}'(t) = \langle 3t^{1/2}, -2\sin(2t), 2\cos(2t)\rangle.$$
   Next, we integrate the magnitude:
   $$\begin{aligned} L &= \int_0^1 |\underline{r}'(t)|\, dt, \\ &= \int_0^1 \sqrt{9t + 4\sin^2(2t) + 4\cos^2(2t)}\, dt, \\ &= \int_0^1 \sqrt{9t+4}\, dt, \\ &= \left[\frac{2}{27}(9t+4)^{3/2}\right]_0^1, \\ &= \frac{2}{27}\left((13)^{3/2} - 4^{3/2}\right). \end{aligned}$$

2. (10 points) (a) Let C be the curve in the xz-plane given by $z = \frac{1}{x}, 2 \leq x \leq 5$. Find parametric equations for the surface S obtained by rotating the curve C around the z-axis.

One possible solution

Let

$$\begin{aligned} x &= u\cos(v), \\ y &= u\sin(v), \\ z &= \frac{1}{u}, \end{aligned}$$

where $u \in [2,5]$ and $v \in [0, 2\pi]$.

(b) Find parametric equations of the upper half of the sphere centered at $(0,0,1)$ and with radius $R = 3$.

One possible solution

Let

$$\begin{aligned} x &= 3\cos(u)\sin(v), \\ y &= 3\sin(u)\sin(v), \\ z &= 3\cos(v) + 1, \end{aligned}$$

where $u \in [0, 2\pi]$ and $v \in [0, \frac{\pi}{2}]$.

3. (20 points) Let $z = f(x,y) = e^{-(x^2+y^2)}$ model a mountain.

(a) If a hiker standing at $(\frac{1}{2}, \frac{1}{3})$ wishes to descend as quickly as possible, in what direction must she walk?

Solution

The gradient of f will point in the direction of quickest increase, so we want

$$-\nabla f = \langle 2xe^{-(x^2+y^2)}, 2ye^{-(x^2+y^2)} \rangle$$

evaluated at $(\frac{1}{2}, \frac{1}{3})$:

$$\left\langle e^{-(\frac{1}{4}+\frac{1}{9})}, \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})} \right\rangle.$$

(b) How steep is the slope from $(\frac{1}{2}, \frac{1}{3})$ in the direction of $\underline{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. This means that you have to find the directional derivative $z = f(x,y)$ in the $\underline{u}$ direction.

Solution

We may use the gradient to calculate the directional derivative:

$$D_{\underline{u}}f = \nabla f \cdot \underline{u} = \frac{5\sqrt{2}}{6}e^{-(\frac{1}{4}+\frac{1}{9})}.$$

(c) Find an equation of the tangent plane to $z = f(x,y) = e^{-(x^2+y^2)}$ at the point $(\frac{1}{2}, \frac{1}{3})$ where the hiker is standing.

Solution

We may represent the surface as the level curve

$$F(x,y,z) = e^{-(x^2+y^2)} - z = 0.$$

The gradient of F evaluated at $(\frac{1}{2}, \frac{1}{3}, e^{-(\frac{1}{4}+\frac{1}{9})})$ is the normal vector for the tangent plane:

$$\nabla F = \left\langle e^{-(\frac{1}{4}+\frac{1}{9})}, \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})}, -1 \right\rangle.$$

An equation for the plane is therefore

$$e^{-(\frac{1}{4}+\frac{1}{9})}\left(x - \frac{1}{2}\right) + \frac{2}{3}e^{-(\frac{1}{4}+\frac{1}{9})}\left(y - \frac{1}{3}\right) - \left(z - e^{-(\frac{1}{4}+\frac{1}{9})}\right) = 0.$$

4. (20 points) (a) Determine whether the following limit exists. If the limit exists, find it. Explain your answer.

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2-2y^3}\right)$$

Solution
Using polar coordinates, we see that

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2-2y^3}\right) &= \lim_{r\to 0}\left(\frac{r^2\cos(\theta)\sin(\theta)}{r^2\cos^2(\theta)-2r^3\sin^3(\theta)}\right), \\ &= \lim_{r\to 0}\left(\frac{\cos(\theta)\sin(\theta)}{\cos^2(\theta)-2r\sin^3(\theta)}\right), \\ &= \tan(\theta).\end{aligned}$$

The limit depends on θ and therefore does not exist.

(b) Is the following function continuous at $(0,0)$? Use limits to explain your answer.

$$f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0), \\ 1 & \text{if } (x,y)=(0,0), \end{cases}$$

Solution
The function is continuous at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0).$$

Again using polar coordinates, we have

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0}\frac{\sin(r^2)}{r^2}, \\ &= \lim_{r\to 0}\frac{2r\cos(r^2)}{2r}, \text{ by L'Hôpital's Rule} \\ &= \lim_{r\to 0}\cos(r^2) = 1.\end{aligned}$$

(c) Calculate $\frac{dz}{dt}$ at $t = \pi$ for $z = f(x, y) = x^2 - xy - 4y^2$, $x(t) = \cos(2t)$, $y(t) = \sin(2t)$.
Solution
First, note that when $t = \pi$, $x = 1$ and $y = 0$. The chain rule tells us that

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}, \\
&= (2x(t) - y(t)) \cdot (-2\sin(2t)) + (-x(t) - 8y(t)) \cdot (2\cos(2t)), \\
&= 0 + (-1 - 0) \cdot 2, \\
&= -2.
\end{aligned}$$

(d) Calculate $\frac{\partial z}{\partial u}$ for $z = f(x, y) = \ln(\frac{x}{y+1})$, $x(u, v) = uv$, $y(u, v) = \frac{u}{v}$.
Solution

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \\
&= \frac{1}{x(u, v)} \cdot v + \frac{-1}{y(u, v) + 1} \cdot \frac{1}{v}, \\
&= \frac{1}{u} - \frac{1}{u + v}.
\end{aligned}$$

5. (20 points) Consider the surface given by the equation

$$x^2 - y^2 + z^2 = 4.$$

(a) Find a vector normal to the surface at $(-1, 1, 2)$.

Solution

Because the surface is a level surface:

$$f(x, y, z) = x^2 - y^2 + z^2 = 4,$$

the gradient vector suffices:

$$\nabla f = \langle 2x, -2y, 2z \rangle.$$

Evaluating at $(-1, 1, 2)$, we have our normal vector:

$$\langle -2, -2, 4 \rangle.$$

(b) Find an equation for the tangent plane to the surface at $(-1, 1, 2)$.

Solution

We use the normal vector previously calculated:

$$-2(x + 1) - 2(y - 1) + 4(z - 2) = 0.$$

6. (15 points) Consider the function $f(x,y) = x^3 + y^2 - 3x - 2y$.

   (a) Find and classify all critical points for the function
   Solution
   First, the relevant partial derivatives:

   $$f_x(x,y) = 3x^2-3, \quad f_y(x,y) = 2y-2, \quad f_{xx}(x,y) = 6x, \quad f_{yy}(x,y) = 2, \quad f_{xy}(x,y) = f_{yx}(x,y) = 0.$$

   Next, the determinant:
   $$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 12x.$$
   Now we find our critical points. The condition that $f_y = 0$ implies
   $$y = 1.$$
   The condition that $f_x = 0$ implies
   $$x = \pm 1.$$
   Thus, we need to classify two points: $(-1,1)$ and $(1,1)$. Evaluating D:
   $$D(-1,1) = -12 < 0, \quad D(1,1) = 12 > 0,$$
   and so $(-1,1)$ is a saddle point. To complete the classification of $(1,1)$, we check the sign of f_{xx}:
   $$f_{xx}(1,1) = 6 > 0,$$
   and so $(1,1)$ is a minimum.

(b) Find the absolute maximum and minimum of $f(x,y)$ in the rectangle with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$.

Solution

We must check the value of f at the corners of the box, check for extrema on the boundary, and finally check for local extrema in the center.

$$\begin{aligned} f(x,0) &= x^3 - 3x, & f'(x,0) &= 3x^2 - 3, \\ f(x,1) &= x^3 - 3x - 1, & f'(x,1) &= 3x^2 - 3, \\ f(0,y) &= y^2 - 2y, & f'(0,y) &= 2y - 2, \\ f(1,y) &= -2 + y^2 - 2y, & f'(1,y) &= 2y - 2. \end{aligned}$$

The list of values we must check is therefore $(0,0),(0,1),(1,1),(1,0)$. As

$$\begin{aligned} f(0,0) &= 0, \\ f(1,0) &= -2, \\ f(1,1) &= -3, \\ f(0,1) &= -1, \end{aligned}$$

we conclude that the maximum is 0 at $(0,0)$ and the minimum is -3 at $(1,1)$.