CALCULUS 3

March2, 2011

2nd TEST

YOUR NAME: **T H E S O L U T I O N S**

- ◯ **001** J. KELLER .... (9AM)
- ◯ **002** B. PURKIS .... (10AM)
- ◯ **003** A. SPINA .... (11AM)
- ◯ **004** A. SPINA .... (12PM)
- ◯ **005** M. NOYES .... (1PM)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then **the wrong one** will be **counted** for your grading!

Make sure you write an arrow on top of vector quantities to differentiate them from scalar quantities (numbers). Remember that, within the same context, $\vec{v}$ (with the arrow) is a *vector* and v (without the arrow) is the *norm* of the previous vector If a vector is the null vector, write an arrow on top of the zero!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	15 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	20 pts	
6	20 pts	
TOTAL	100 pts	

1. **[15 pts]** For following pair of functions f and g, determine if the level curves of the functions cross at right angles, and find their gradients at the point $(1,4)$.

$$f(x,y) = 5x + 5y\,, \qquad g(x,y) = 5x - 5y\,;$$

SOLUTION

At points (x,y) where the gradients are defined and are not the zero vector, the level curves of f and g intersect at right angles if and only $\nabla f \cdot \nabla g = 0$.

We have

$$\nabla f \cdot \nabla g = (5\mathbf{i} + 5\mathbf{j}) \cdot (5\mathbf{i} - 5\mathbf{j}) = 0\,,$$

which is zero at all points, $\boxed{\text{so the level curves of } f \text{ intersect those of } g \text{ in right angles.}}$ Then, we have the gradients at $(1,4)$ already:

$$\boxed{\nabla f(1,4) = 5\mathbf{i} + 5\mathbf{j} \qquad \text{and} \qquad \nabla g(1,4) = 5\mathbf{i} - 5\mathbf{j}\,.}$$

2. **[15 pts]** Check that the point $(-1,-1,3)$ lies on the surface $x^2 - 4y^2 + 2z^2 = 15$. Then, viewing the surface as a level surface for a function $f(x,y,z)$, find a vector normal to the surface and an equation for the tangent plane to the surface at $(-1,-1,3)$.

SOLUTION

First, we check that $\boxed{(-1)^2 - 4(-1)^2 + 2(3)^2 = 15}$

Then let

$$f(x,y,z) = x^2 - 4y^2 + 2z^2$$

so that the given surface is the level surface $f(x,y,z) = 15$. Since

$$f_x = 2x\,, \qquad f_y = (-8)y\,, \qquad \text{and} \qquad f_z = 4z\,,$$

we have

$$\nabla f(-1,-1,3) = -2\,\mathbf{i} + 8\,\mathbf{j} + 12\,\mathbf{k}\,.$$

Since gradients are perpendicular to level surfaces, a vector normal to the surface at $(-1,-1,3)$ is

$$\boxed{\mathbf{n} = -2\mathbf{i} + 8\mathbf{j} + 12\mathbf{k}} \qquad \text{or} \qquad \boxed{\mathbf{n} = -\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}}$$

(there is no need of normalizing this vector). Thus, an equation for the tangent plane is

$$\boxed{-2\,(x+1) + 8\,(y+1) + 12\,(z-3) = 0} \qquad \text{or} \qquad \boxed{-\,(x+1) + 4\,(y+1) + 6\,(z-3) = 0}$$

or

$$\boxed{-x + 4y + 6z = 15}$$

3. **[15 pts]** If

$$z = xe^y\,, \qquad x = u^2 + v^2\,, \qquad y = u^2 - v^2\,,$$

<u>use the chain rule</u> find $\partial z/\partial u$ and $\partial z/\partial v$.

SOLUTION

Since z is a function of two variables x and y which are functions of two variables u and v, the two chain rule identities which apply are:

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= (e^y)(2u) + (xe^y)(2u) \\
&= \boxed{2u\,(1 + u^2 + v^2)\,e^{(u^2-v^2)}} \\
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= (e^y)(2v) + (xe^y)(-2v) \\
&= \boxed{2v\,(1 - x^2 - y^2)\,e^{(u^2-v^2)}}
\end{aligned}$$

4. **[15 pts]** Find the linear, $L(x,y)$, and quadratic, $Q(x,y)$, Taylor polynomials valid near $(0,2)$ for

$$f(x,y) = \cos(x)\sin(y-2)\,.$$

SOLUTION

1st Way. We have $f(0,2) = 0$ and the relevant derivatives are:

$$\begin{array}{rclcrcl} f_x &=& -\sin(x)\ \sin(y-2) &\Rightarrow& f_x(0,2) &=& 0\,, \\ f_y &=& \cos(x)\ \cos(y-2) &\Rightarrow& f_y(0,2) &=& 1\,, \\ f_{xx} &=& -\cos(x)\ \sin(y-2) &\Rightarrow& f_{xx}(0,2) &=& 0\,, \\ f_{xy} &=& -\sin(x)\ \cos(y-2) &\Rightarrow& f_{xy}(0,2) &=& 0\,, \\ f_{yy} &=& -\cos(x)\ \sin(y-2) &\Rightarrow& f_{yy}(0,2) &=& 0\,. \end{array}$$

Thus the linear approximation, $L(x,y)$ to $f(x,y)$ at $(0,2)$, is given by:

$$f(x,y) \;\approx\; L(x,y) = f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2) = y-2\,.$$

The quadratic approximation, $Q(x,y)$ to $f(x,y)$ near $(0,2)$, is given by:

$$\begin{aligned} f(x,y) \;\approx\; Q(x,y) \;&=\; f(0,2) + f_x(0,2)(x-0) + f_y(0,2)(y-2) + \\ &\quad \frac{1}{2} f_{xx}(0,2)(x-0)^2 + f_{xy}(0,2)(x-0)(y-2) + \frac{1}{2} f_{yy}(0,2)(y-2)^2 \\ &=\; y-2\,. \end{aligned}$$

2nd Way. Using one-variable Taylor expansion we get

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2} + \text{terms of order } x^4 \text{ and higher}, \\ \sin(y-2) &= (y-2) + \text{terms of order } (y-2)^3 \text{ and higher}, \end{aligned}$$

then,

$$\begin{aligned} \cos(x)\sin(y-2) &= \left(1 - \frac{x^2}{2} + \cdots\right)\left((y-2) + \cdots\right) \\ &= (y-2) + \text{terms of order } \textit{higher} \text{ than 2 in } x \text{ and } (y-2). \end{aligned}$$

Hence,

$$\boxed{\begin{array}{rcl} L(x,y) &=& y-2 \\ Q(x,y) &=& y-2 \end{array}}$$

5. **[20 pts]** Find the critical points for the function

$$f(x,y) = 32xy - (x+y)^4$$

and classify each as a local maximum, local minimum, saddle point, or none of these.

SOLUTION

At a critical point, $f_x(x_c,y_c) = 0$, and $f_y(x_c,y_c) = 0$. Thus

$$\begin{array}{lcll} & & f_x = 32y - 4(x+y)^3 = 0 & (1) \\ & & f_y = 32x - 4(x+y)^3 = 0 & (2) \\ (1\ \&\ 2) & \Rightarrow & 32y = 4(x+y)^3 = 32x & (3) \\ (3) & \Rightarrow & x = y & (4) \\ (1\ \&\ 4) & \Rightarrow & 32x + 32x^3 = 0 & (5) \\ (5) & \Rightarrow & x = 0 \quad \text{or} \quad x = \pm 1 & (6) \\ (4\ \&\ 6) & \Rightarrow & \boxed{(x_c,y_c) = (0,0)\,,\ \ (1,1)\,,\ \ (-1,-1)} & \end{array}$$

To classify the critical points we need to compute the discriminant

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y)\,.$$

The second partial derivatives are,

$$\begin{aligned} f_{yy}(x,y) &= -12(x+y)^2\,, \\ f_{xx}(x,y) &= -12(x+y)^2\,, \\ f_{xy}(x,y) &= 32 - 12(x+y)^2\,, \end{aligned}$$

so

$$D(x,y) = 144(x+y)^4 - \left(32 - 12(x+y)^2\right)^2\,.$$

Thus

at $(0,0)$	$D(0,0) = -32^2 < 0$	$\Rightarrow$	saddle point
at $(1,1)$	$D(1,1) = 144 \times 2^4 - (32 - 12 \times 4)^2 = 2048 > 0$		
	& $f_x(1,1) = -48 < 0$	$\Rightarrow$	local maximum
at $(-1,-1)$	$D(-1,-1) = 144 \times 2^4 - (32 - 12 \times 4)^2 = 2048 > 0$		
	& $f_x(-1,-1) = -48 < 0$	$\Rightarrow$	local maximum

6. **[20 pts]** Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 2x - y$ subject to the constraint $x^2 + 3y^2 = 39$, if such values exist.

SOLUTION

Our *objective function* is $\boxed{f(x,y) = 2x - y}$ and our *equation of constraint* is $\boxed{g(x,y) = x^2 + 3y^2 - 39}$
To optimize $f(x,y)$ with Lagrange multipliers we solve the following system of equations

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) && (1) \\ f_y(x,y) &= \lambda g_y(x,y) && (2) \\ g(x,y) &= 0 && (3) \end{aligned}$$

$$\begin{array}{lcll} (1) & \Rightarrow & 2 = 2\lambda x & (4) \\ (2) & \Rightarrow & -1 = 6\lambda y & (5) \\ (3) & \Rightarrow & x^2 + 3y^2 = 39 & (6) \\ (4) & \Rightarrow & \lambda = 1/x & (7) \\ (5) & \Rightarrow & \lambda = -1/(6y) & (8) \\ (7\ \&\ 8) & \Rightarrow & x = -6y & (9) \\ (6\ \&\ 9) & \Rightarrow & (-6y)^2 + 3y^2 = 39 & (10) \\ (10) & \Rightarrow & y = \pm 1 & (11) \\ (9\ \&\ 11) & \Rightarrow & \boxed{(x_c, y_c) = (-6,+1)\,,\ \ (+6,-1)} & (12) \end{array}$$

Since the constraint is closed and bounded, maximum and minimum values of f subject to the constraint exist. Evaluating f at the critical points, we find

at $(-6,+1)$	$f(+6,-1) = 2(+6) + (+1) = +13$	maximum
at $(+6,-1)$	$f(-6,+1) = 2(-6) + (-1) = -13$	minimum