1. (20) Consider the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ where

$$\mathbf{r}(t) \ = \ \langle 3 + 3\cos t, 3 + 4t, 12 + 3\sin t \rangle, \ 0 \le t \ \le 2\pi.$$

(i) Find the velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$ and a formula for the unit tangent vector $\mathbf{T}(t)$ of the curve.

(ii) Find a formula for the curvature $\kappa(t)$ of the curve at the point $\mathbf{r}(t)$.

2. (10) Find the dimensions that minimize the cost of the material needed to construct a rectangular box with a volume of 12m^3 if the material for its bottom costs twice as much per square meter as the material for its top and four sides.

3. (10) A rectangular block has dimensions $x = 3$ meters, $y = 4$ meters and $z = 1$ meter. If x is increasing at 1 m/s, y is increasing at 2 m/s and z is decreasing at 2 m/s, use the chain rule to find the rate of change of the volume of the block.

4. (10) Consider the unit sphere, $x^2 + y^2 + z^2 = 1$.

(i) Write the sphere as $F(x, y, z) = 0$, and find the gradient vector ∇F.

(ii) Use the gradient vector to find the tangent plane to the sphere at the point

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

5. (20) Let R be the region on the xy-plane that is bounded by

$$y = \frac{1}{x}, \quad y = \frac{4}{x}$$

and

$$x = \frac{1}{y^2}, \quad x = \frac{8}{y^2}.$$

Using change of variables, or otherwise, evaluate the integral

$$\iint_R xy \, dx \, dy.$$

6. (15) Calculate the line integral

$$\int_C f(x,y)\,ds,$$

where $f(x,y) = x$ and C is the part of the graph of $x = y^3$ from $(-1,-1)$ to $(8,2)$.

7. (20)

(i) Find a polar equation for the line passing through $(1,0)$ and $(0,1)$ on the xy-plane.

(ii) Find the volume of the solid under the surface

$$z = \frac{\sqrt{2}}{\sqrt{x^2 + y^2}}$$

and above the smaller region on the xy-plane bounded by the circle $x^2 + y^2 = 1$ and the line passing through $(1, 0)$ and $(0, 1)$.

Hint: You may want to make use of (i) and the identities
$\sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \pi/4)$ and $\int \csc x dx = -\ln|\csc x + \cot x|$.

8. (15) The vector field

$$\mathbf{F}(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

is conservative.

(i) Find a potential function for $\mathbf{F}$.

(ii) Now use the potential function to evaluate

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds,$$

where $\mathbf{F}$ is as above, and C is some curve from the point $(1, 2, 3)$ to the point $(-1, 4, 2)$.

9. (15) Let C denote the piecewise smooth simple closed curve that is the perimeter of the square $[0,1] \times [0,1]$, oriented in the counterclockwise direction. Let $\mathbf{F}(x,y)$ be the vector field in $\mathbb{R}^2$ defined by $\mathbf{F}(x,y) = \langle x^3 - y^2, x^4 \rangle$.

Evaluate the line integral

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds,$$

i.e., evaluate the line integral

$$\oint_C (x^3 - y^2) dx + x^4 dy.$$

10. (20) The Divergence Theorem as stated in the book tells us (with the proper suppositions) that

$$\iint_S \mathbf{F}.\mathbf{n}\, dS = \iiint_T \nabla.\mathbf{F}\, dV,$$

where S is the closed, piecewise smooth surface that bounds the space region T. Evaluate both sides of this equation for the field $\mathbf{F} = \langle x, y, z \rangle$ over the sphere $x^2 + y^2 + z^2 = a^2$.

11. (25) Use Stokes' Theorem to evaluate

$$\oint \mathbf{F}.\mathbf{T}\, ds,$$

where $\mathbf{F} = \langle y, xz, x^2 \rangle$ and C is the boundary of the triangle cut from the plane $x+y+z=1$ by the first octant, counterclockwise when viewed from above.

Name: ______________________________

Section: ______________________________

University of Colorado

Mathematics 2400: Final Examination

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Justify all your answers!

Problem	Points	Score
1	20	
2	10	
3	10	
4	10	
5	20	
6	15	
7	20	
8	15	
9	15	
10	20	
11	25	
Total	180	