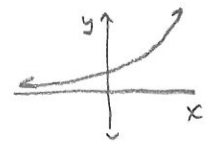
WARMUP #10 - END BEHAVIOR ($\lim_{x\to\infty} f(x)$)

1. BASIC LIMITS

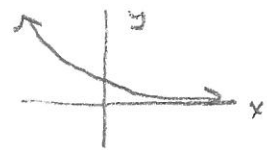
$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to -\infty} e^x = 0$



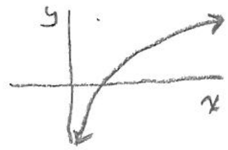
$\lim_{x\to\infty} e^{-x} = 0$

$\lim_{x\to -\infty} e^{-x} = \infty$



$\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to 0^+} \ln x = -\infty$



$\lim_{x\to\infty} \arctan x = \pi/2$

$\lim_{x\to -\infty} \arctan x = -\pi/2$

FOR $n>0$: $\lim_{x\to\infty} x^n = \infty$, $\lim_{x\to\infty} \frac{1}{x^n} = 0$

2. BEGINNER EXAMPLES

A. $\lim_{x\to\infty} e^{\frac{1}{x}} =$

B. $\lim_{x\to\infty} \frac{1}{\ln x} =$

C. $\lim_{x\to\infty} \ln\left(\frac{1}{x}\right) =$

D. $\lim_{x\to\infty} \sin(\arctan x) =$

ANSWERS: A: 1, B: 0, C: $-\infty$, D: 1

3. INDETERMINATE FORMS:

"$\frac{\infty}{\infty}$", "$\frac{0}{0}$", "$0\cdot\infty$", "$\infty-\infty$", "∞^0", "1^∞", "0^0"

4. USE YOUR INTUITION TO GUESS:

EX $\lim_{x\to\infty} \frac{x^2+3x}{x^3+2x} = 0$ SINCE x^3 GROWS FASTER THAN x^2

EX $\lim_{x\to\infty} \frac{3x^4-1}{2x^4-x} = \frac{3}{2}$, POWERS ARE THE SAME, LIMIT IS RATIO OF LEADING COEFFICIENTS

EX $\lim_{x\to\infty} \frac{e^x}{x^5} = \infty$ SINCE e^x GROWS FASTER THAN x^5

EX $\lim_{x\to\infty} \frac{x-x^5}{x^2+1} = -\infty$ SINCE x^5 GROWS FASTER THAN x^2

EX $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$ SINCE $\sqrt{x}$ GROWS FASTER THAN $\ln x$

EX $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{3x} = \frac{1}{3}$ SINCE $\sqrt{x^2+1}$ IS ROUGHLY $\sqrt{x^2}=x$, THE POWERS ARE THE SAME

5. USE YOUR TOOLS TO RIGOROUSLY CONFIRM YOUR GUESS:

- ALGEBRA: DIVIDE NUMERATOR AND DENOMINATOR BY DOMINANT TERM IN DENOMINATOR:

EX
$$\lim_{x \to \infty} \frac{2x^{48} + 3x^2 - 1}{x^{48} - x + 1}$$

$$= \lim_{x \to \infty} \frac{2x^{48} + 3x^2 - 1}{x^{48} - x + 1} \cdot \frac{\frac{1}{x^{48}}}{\frac{1}{x^{48}}}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{3}{x^{46}} - \frac{1}{x^{48}}}{1 - \frac{1}{x^{47}} + \frac{1}{x^{48}}} = \frac{2}{1}$$

EX
$$\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{3x}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2+1}}{3x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2+1}\sqrt{\frac{1}{x^2}}}{3x \cdot \frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{1 + \frac{1}{x^2}}}{3} = \frac{1}{3}$$

- L'Hôpital's rule: USE FOR FORMS "$\frac{0}{0}$" OR "$\frac{\infty}{\infty}$"

EX
$$\lim_{x \to \infty} \frac{x^2}{e^x} \qquad \text{FORM } \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{\text{L'Hô}}{=} \lim_{x \to \infty} \frac{2x}{e^x} \qquad \text{FORM } \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{\text{L'Hô}}{=} \lim_{x \to \infty} \frac{2}{e^x} = 0$$

- FOR LIMITS OF THE FORM "$0 \cdot \infty$", REWRITE AS A QUOTIENT, THEN L'HÔ.

EX
$$\lim_{x \to 0^+} x \ln x$$

$$= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \qquad \text{FORM } \text{"}\frac{-\infty}{\infty}\text{"}$$

$$\overset{\text{L'Hô}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-x^{-2}} \qquad \text{SIMPLIFY BEFORE PROCEEDING}$$

$$= \lim_{x \to 0^+} -x = 0$$

- EXPONENTIAL INDETERMINATE FORMS: USE LOGS, THEN TOOLS FOR "$0 \cdot \infty$", THEN L'HÔPITAL'S. EXPONENTIATE AT THE END.

EX: $\lim_{x \to 0^+} x^{\sin x}$

$L = \lim_{x \to 0^+} x^{\sin x}$ (GIVE A NAME TO THE LIMIT)

$\ln L = \ln\left(\lim_{x \to 0^+} x^{\sin x}\right)$ (TAKE NATURAL LOG OF EACH SIDE)

$= \lim_{x \to 0^+} \ln\left(x^{\sin x}\right)$ (BECAUSE $\ln x$ IS CONTINUOUS)

$= \lim_{x \to 0^+} \sin x \ln(x)$ (LAWS OF LOGS)

$= \lim_{x \to 0^+} \frac{\ln x}{\csc x}$ (REWRITE AS QUOTIENT, NOW IT IS OF THE FORM "$\frac{-\infty}{+\infty}$")

$\overset{\text{L'H}\hat{\text{O}}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}$ (L'Hôpital's rule)

$= \left(\lim_{x \to 0^+} \frac{\sin^2 x}{x}\right)\left(\lim_{x \to 0^+} \frac{1}{\cos x}\right)$ (Rewrite, splitting off easy part)

$= \lim_{x \to 0^+} \frac{\sin^2 x}{x}$ (FORM IS NOW "$\frac{0}{0}$")

$\overset{\text{L'H}\hat{\text{O}}}{=} \lim_{x \to 0^+} \frac{2 \sin x \cos x}{1} = 0$

Now we have $\ln L = 0$, So $L = e^0 = 1$

- SQUEEZE LAW (ALSO KNOWN AS "SANDWICH THEOREM")

If $f(x) \leq g(x) \leq h(x)$ AND $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L$,

THEN $\lim_{x \to \infty} g(x) = L$

EX: $\lim_{x \to \infty} \frac{\sin x}{e^x}$

Since $-1 \leq \sin x \leq 1$, $-\frac{1}{e^x} \leq \frac{\sin x}{e^x} \leq \frac{1}{e^x}$

$\lim_{x \to \infty} \frac{1}{e^x} = 0$ AND $\lim_{x \to \infty} -\frac{1}{e^x} = 0$. So BY THE SQUEEZE LAW, $\lim_{x \to \infty} \frac{\sin x}{e^x} = 0$.

PRACTICE PROBLEMS

I. USING ONLY YOUR INTUITION, GUESS THE FOLLOWING LIMITS:

A. $\lim_{x \to \infty} \dfrac{e^x}{x^2}$

B. $\lim_{x \to \infty} \ln\left(\dfrac{x}{x+1}\right)$

C. $\lim_{x \to \infty} e^{\frac{1-x^3}{x}}$

D. $\lim_{x \to \infty} \ln\left(\dfrac{3x}{x^2+1}\right)$

E. $\lim_{x \to \infty} \dfrac{\sin\left(\frac{1}{x}\right)}{\arctan x}$

F. $\lim_{x \to \infty} \dfrac{\ln x}{n^5}$

G. $\lim_{x \to \infty} \dfrac{\sqrt[3]{x}}{e^x \ln x}$

H $\lim_{x \to \infty} \dfrac{\sqrt{x^7+x}}{\sqrt[3]{x^{10}-x}}$

II. USE YOUR TOOLS TO CALCULATE THESE LIMITS

A. $\lim_{x \to \infty} \dfrac{\sqrt{3x^2+4}}{x-1}$

B. $\lim_{x \to \infty} \dfrac{e^{2x}}{e^{2x}+e^x}$

C. $\lim_{x \to \infty} \dfrac{\sqrt[3]{x}}{\ln x}$

D. $\lim_{x \to \infty} x^2 e^{-x^2}$

E. $\lim_{x \to \infty} x \tan\left(\frac{1}{x}\right)$

F. $\lim_{x \to \infty} x^{\frac{1}{x}}$

G. $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$

H. $\lim_{x \to \infty} \dfrac{x \sin^2 x}{e^{2x}}$