Integral Practice Problems (Provided by Patrick Wynne)
Evaluate the following integrals. Note: The last two pages are significantly more challenging.

1. $\int e^{\sqrt{x}}\,dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$. (Use integration by parts with $u = 2\sqrt{x}$ and $v = e^{\sqrt{x}}$.)

2. $\int \frac{\cos x}{\sqrt{2-\sin^2 x}}\,dx = \sin^{-1}\left(\frac{\sin x}{\sqrt{2}}\right) + C$. (Do u substitution with $u = \sin x$.)

3. $\int \frac{1}{x\ln x}\,dx = \ln(\ln x) + C$. (Do u substitution with $u = \ln x$.)

4. $\int \frac{x^5 + \sqrt[3]{x} - 1}{x^{\frac{4}{3}}}\,dx = \frac{3}{14}x^{14/3} + \ln|x| + 3x^{-1/3} + C$. (Simplify to $x^{11/13} + x^{-1} - x^{-4/3}$ then use the power rule.)

5. $\int x\sec x\tan x\,dx = x\sec x - \ln|\sec x + \tan x| + C$. (Use integration by parts with $u = x$ and $dv = \sec x\tan x dx$.)

6. $\int \frac{x}{\sqrt{x-1}}\,dx = \frac{2}{3}(x-1)^{3/2} + 2\sqrt{x-1} + C$. (Do u substitution with $u = x-1$, so $x = u+1$.)

7. $\int \frac{\sec x \tan x}{1+\sec x}\,dx = \ln|1+\sec x|$. (Do u substitution with $u = 1+\sec x$).

8. $\int x \tan^{-1} x\,dx = \frac{1}{2}((x^2+1)\arctan x - x) + C$. (Use Integration by Parts with $u = \arctan x$ and $dv = xdx$).

9. $\int \frac{5x+1}{x^3-x}\,dx = \ln|\frac{(x-1)^3}{x(x+1)^2}|$. (Use Partial Fractions decomposition then simplify the log).

10. $\int x\sin(5x^2+2)\,dx = -1/10\cos(5x^2+2) + C$. (Do u substitution with $u = 5x^2+2$).

11. $\int \left(\sin^{-1} x\right)^2 dx = 2\sqrt{1-x^2}\sin^{-1}(x) - 2x + x\left(\sin^{-1} x\right)^2 + C$. (Do integration by parts with $u = \sin^{-1}(x)$ and $dv = \sin^{-1}(x)dx$).

12. $\int x\sin^{-1} x\, dx = \frac{1}{4}\left(x\sqrt{1-x^2} + (2x^2-1)\sin^{-1} x + C.\right)$ (Do integration by parts with $u = x$ and $dv = \sin^{-1} x dx$).

13. $\int \sin^{-1}\sqrt{x}\, dx = \frac{1}{2}\left(\sqrt{x-x^2} + (2x-1)\sin^{-1}\sqrt{x}\right)$. (Do u substitution with $u = \sqrt{x}$, so $dx = 2udu$ then follow the process from the preceding problem).

14. $\int \frac{1}{1-\tan^2 x}\, dx = \frac{1}{4}\left(2x - \ln(\cos x - \sin x) + \ln(\cos x + \sin x)\right)$. (Do u substitution with $u = \tan x$, so $dx = \frac{du}{1+u^2}$, using the identity $\tan^2(x) + 1 = \sec^2(x)$. Then use partial fractions.)

15. $\int \ln\left(\sqrt{x-1} + \sqrt{x+1}\right)\, dx = \frac{1}{2}\left(x\ln 2 + x\ln(x+\sqrt{x^2-1}) - \sqrt{x^2-1}\right)$. (Notice that $\sqrt{x-1} + \sqrt{x+1} = \sqrt{2x + 2\sqrt{x^2-1}}$. Use this to simplify the logarithm then do integration by parts with $u = \ln(x+\sqrt{x^2-1})$ and $dv = dx$).

16. $\displaystyle\int \frac{1}{x-\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(\ln|x-\sqrt{1-x^2}| - \sin^{-1}(x)\right) + C$. (Do a trigonometric substitution with $x = \sin\theta$, so $dx = \cos\theta d\theta$ then rewrite $\cos\theta$ as $\frac{1}{2}\left((\cos\theta - \sin\theta) + (\cos\theta + \sin\theta)\right)$. Then split it in two and do a u substitution with $u = \sin\theta - \cos\theta$).

17. $\displaystyle\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}\,dx = \frac{2}{3}\sqrt{3e^{2x}-6e^x-1} + \frac{1}{\sqrt{3}}\ln|3-3e^x - \sqrt{9e^{2x}-18e^x-3} + C$. (Do a u substitution with $u = e^x$ then complete the square in the denominator, i.e. $3u^2-6u-1 = (\sqrt{3}u - \sqrt{3})^2 - 4$. Then do another substitution with $v = \sqrt{3}u - \sqrt{3}$. Split it into two integrals, one with constant numerator and one with numerator equal to cv for some constant c. Then one more substitution gives the desired result.

18. $\displaystyle\int \frac{1}{x^6-1}\,dx = \frac{1}{12}(\ln(x^2-x+1) - \ln(x^2+x+1) + 2\ln|x-1| - 2\ln|x+1|)$
$-2\sqrt{3}\left(\tan^{-1}(\frac{2x-1}{\sqrt{3}}) - \tan^{-1}(\frac{2x+1}{\sqrt{3}})\right)$.
(Factor the denominator as $x^6-1 = (x-1)(x+1)(x^2-x+1)(x^2+x+1)$ and note that the two quadratic terms are irreducible over the reals. Use the partial fractions decomposition to rewrite the integral. Once the coefficients are determined, each piece is straightforward to integrate).

19. $\displaystyle\int \frac{1}{x^4+4}\,dx = \frac{1}{16}\left(\ln(x^2+2x+2) - \ln(x^2-2x+2) - 2\tan^{-1}(1-x) + 2\tan^{-1}(1+x)\right)$. (Factor the denominator as $x^4+4 = (x^2+2x+2)(x^2-2x+2)$ and note that each of the two factors are irreducible over the reals. Then use the partial fractions decomposition).

20. $\displaystyle\int \frac{1}{x(x+1)(x+2)\cdots(x+n)}\,dx = a_0\ln|x| + a_1\ln|x+1| + \cdots + a_n\ln|x+n|$,
where $a_i = 1/(-1)^i(i)!(n-i)!$ for $i = 1,\dots,n$.. (Use the partial fractions decomposition to write as $a_0/x + a_1/(x+1) + \cdots + a_n/(x+n)$ for some constants $a_0, a_1, \dots, a_n$. The i^{th} term integrates to $a_i\ln|x+i|$ for all $i = 1,\dots,n$. To solve for the coefficients, write the decomposed expression with a common denominator. Notice that taking $x = i$ makes every term zero except the one involving a_i, which equals $a_i(-1)^i(i)!(n-i)!$ and this equals the expression on the other side evaluated at i, which is always 1. Hence, $a_i = 1/(-1)^i(i)!(n-i)!$ for all $i = 1,\dots,n$).