Series - summing it all up

Here's a list of all of the convergence tests for series that you know so far:

- Divergence test (a.k.a. n-th term test)
- Geometric series test
- Integral test
- p-series
- Term-size comparison test (your book calls this the "comparison test")
- Limit comparison test
- Alternating series test
- Error bounds for alternating series
- Absolute convergence implies convergence
- The Ratio test

Here are the details:

The Divergence test: When you're given a series $\sum_{n=1}^{\infty} a_n$, check the limit of the underlying sequence. If $\lim_{n\to\infty} a_n \neq 0$, then you can conclude that the given series $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, you can't conclude anything yet and you have to do more work.

Geometric series test: You can recognize a geometric series because it is built from an exponential sequence. An infinite geometric series generally has the form $\displaystyle\sum_{n=1}^{\infty} ax^{n-1}$. An infinite geometric series diverges if $|x| \geq 1$, and converges if $|x| < 1$. If an infinite geometric series converges, it converges to a sum of $\dfrac{a}{1-x}$.
(Finite geometric series always converge, don't forget we have a special formula for their sum.)

The Integral test: Generally, this is our last resort, because in order to use it we have to evaluate the corresponding improper integral. Also, we have to make sure the corresponding function is decreasing and positive
Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.

- If $\int_1^{\infty} f(x)dx$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
- If $\int_1^{\infty} f(x)dx$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

***p*-series:** The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Term-size comparison test: (for positive term series)

Suppose that $0 \leq a_n \leq b_n$ for all n beyond a certain value.

- If $\sum b_n$ converges, then $\sum a_n$ converges.
- If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Limit comparison test: (for positive term series)
Suppose $a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is finite and not zero, then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Alternating series test:
The terms in an alternating series alternative signs. They have the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ (if the first term is positive) or $\sum_{n=1}^{\infty}(-1)^n a_n$ (if the first term is negative), where a_n is nonnegative. If $0 < a_{n+1} < a_n$ for all n (i.e., the sequence is decreasing) and $\lim_{n\to\infty} a_n = 0$, then the series converges.

Error Bounds for alternating series:
If you have an alternating series for which you can use the alternating series test to show convergence, then you can get a bound for how accurately the nth partial sum estimates the sum of the series:
Let $\sum_{i=1}^{n}(-1)^{i-1}a_i$ be the nth partial sum of an alternating series and let $S = \lim_{n\to\infty} S_n$ be the sum of the infinite series. Suppose that $0 < a_{n+1} < a_n$ for all n and that $\lim_{n\to\infty} a_n = 0$. Then

$$|S - S_n| < a_{n+1}$$

Absolute convergence implies convergence:
If we have a series $\sum_{n=1}^{\infty} a_n$ that has a mix of negative and positive terms, but that doesn't necessarily alternate, sometimes it is useful to consider $\sum_{n=1}^{\infty} |a_n|$. Here's why:

If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

We call a series *absolutely convergent* when $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series *conditionally convergent* if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.
So there are three distinct possibilities for a series: it either converges absolutely, converges conditionally, or diverges.

The Ratio test: Suppose you calculate the following limit, and

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $L > 1$ (including if $L = \infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$, we can make no conclusion about the series using this test.

The ratio test is typically useful if a series has geometric components and/or factorial components, possibly mixed with power functions. It is extremely important in determining convergence of power series (see Section 9.5).