MATH 2300 – review problems for Exam 1

1. Evaluate the integral $\int \sin x \cos x \, dx$ in each of the following ways:

   (a) Integrate by parts, with $u = \sin x$ and $dv = \cos x \, dx$. The integral you get on the right should look much like the one you started with, so you can solve for this integral. (Some people call this the "boomerang" method.)

   (b) Integrate by parts, with $u = \cos x$ and $dv = \sin x \, dx$.

   (c) Substitute $w = \sin x$.

   (d) Substitute $w = \cos x$.

   (e) First use the fact that $\sin x \cos x = \frac{1}{2}\sin(2x)$, and then antidifferentiate directly.

   (f) Show that answers to parts (a)–(e) of this problem are all the same. It may help to use the identities $\cos^2 x + \sin^2 x = 1$ and $\cos(2x) = 1 - 2\sin^2 x$.

2. Let $f(x)$ be a continuous function on the set of all real numbers. Show that

$$\int_0^1 f(e^x)\, e^x \, dx = \int_1^e f(x)\, dx.$$

3. (a) Explain why the integral

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2 - 4}}$$

   is improper.

   (b) Show that

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2 - 4}} = \sqrt{21}.$$

4. Suppose that $\displaystyle\int_0^1 f(t) dt = 5$. Calculate the following:

   (a) $\displaystyle\int_0^{0.5} f(2t) dt$

   (b) $\displaystyle\int_0^1 f(1-t) dt$

   (c) $\displaystyle\int_1^{1.5} f(3-2t) dt$

5. Evaluate the following integrals:

   (a) $\displaystyle\int 2x \cos(x^2)\, dx$

   (b) $\displaystyle\int e^{2x} \sin(2x)\, dx$

   (c) $\displaystyle\int \cos^2 \theta \, d\theta$

(d) $\int x^2 \sin(x) dx$

(e) $\int \frac{1}{x^2\sqrt{16-x^2}}\, dx$

(f) $\int \frac{3x^2+6}{x^2(x^2+3)}\, dx$

(g) $\int \sqrt{25-x^2}\, dx$

(h) $\int \frac{3x-1}{x^2-5x+6}\, dx$

(i) $\int \sin^3(5x)\cos(5x)\, dx$

(j) $\int_2^3 \frac{x^2}{1+x^3}\, dx$

(k) $\int_0^3 x\, e^{x^2}\, dx$

(l) $\int x^7 e^{x^4}\, dx$

(m) $\int (\ln(x))^2\, dx$

6. Evaluate the following integrals

   (a) $\int y\sqrt{y^2+1}\, dy$

   (b) $\int y\sqrt{y+1}\, dy$

7. (a) Calculate $\int_2^4 \frac{dx}{(x-3)^2}$, if it exists.

   (b) Find $\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\, dx$, if it converges.

8. Which of the following integrals can be integrated using partial fractions?

   (a) $\int \frac{1}{x^4-5x^2+4}\, dx$

   (b) $\int \frac{1}{x^4+1}\, dx$

   (c) $\int \frac{1}{x^3-8}\, dx$

(d) $\displaystyle\int \frac{2x+1}{x^2+4}\,dx$

9. For this set of problems, state which techniques are useful in evaluating the integral. You may choose from: integration by parts; partial fractions; long division; completing the square; trig substitution; or another substitution. There may be multiple answers.

   (a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

   (b) $\displaystyle\int \frac{1}{\sqrt{6x-x^2-8}}\,dx$

   (c) $\displaystyle\int x\sin x\,dx$

   (d) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

   (e) $\displaystyle\int \frac{x^2}{1-x^2}\,dx$

   (f) $\displaystyle\int (1+x^2)^{-3/2}\,dx$

   (g) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

   (h) $\displaystyle\int \frac{1}{1-x^2}\,dx$

10. Let f be a differentiable function. Suppose that $f''(0)=1$, $f''(1)=2$, $f'(0)=3$, $f'(1)=4$, $f(0)=5$, $f(1)=6$. Compute $\int_0^1 f(x)f'(x)dx$.

11. Determine whether the following improper integrals converge or diverge.

    (a) $\displaystyle\int_1^\infty \frac{\cos^2 x}{\sqrt{x^3}}\,dx$

    (b) $\displaystyle\int_3^\infty \frac{1}{x^2\ln x}\,dx$

    (c) $\displaystyle\int_1^\infty \frac{3+\sin x}{x^2}\,dx$

    (d) $\displaystyle\int_3^\infty \frac{5+2\sin x}{x-2}\,dx$

    (e) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$

    (f) $\displaystyle\int_1^\infty \frac{1}{x\ln x}\,dx$

    (g) $\displaystyle\int_{10}^\infty \frac{1}{x^2-9}\,dx$

12. Find the area of the region enclosed by the graphs $y=-x^2-2x+2$ and $y=1-2x$.

13. Find the area of the region enclosed by the graphs $y = 2x^2$, $y = -4x + 6$ and the x-axis.

14. Using slices parallel to the base, write a definite integral representing the volume of a cone with a height of 10 cm and a base of diameter 6 cm.

15. Consider the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$.

    (a) Sketch the region and find its area

    (b) Sketch the solid obtained by rotating the above region around the x-axis.

    (c) Write an integral that gives the volume of the solid

    (d) Repeat with the same region rotated around the y-axis.

    (e) Repeat with the same region rotated around the line $x = -5$.

    (f) Repeat with the same region rotated around the line $y = 1$.

16. Find the volume of the solid whose base is the region in the xy-plane bounded by the curves $y = x$ and $y = x^2$ and whose cross sections perpendicular to the x-axis are squares with one side in the xy-plane.

17. Do the same thing as the previous problem except with semi-circle cross sections and then again with cross sections that are isosceles triangles of height 3. Notice that the integrals you get are just multiples of the integral from the previous problem. This means you don't have to evaluate them, instead you can just find the appropriate multiple of the answer to the previous problem!

18. Find the volume of the solid of revolution obtained by rotating the region bounded by the graphs of $y = 2x - x^2$ and $y = 0$ about the line y-axis

19. Find the volume of the solid of revolution obtained by rotating the region bounded by the graphs of $y = 2x^2 - x^3$ and $y = 0$ about the line $x = -4$

20. Suppose $f(0) = 1$, $f(1) = e$, and $f'(x) = f(x)$ for all x. Find

$$\int_0^1 e^x f'(x)dx.$$