1. Evaluate the following integral:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

Let $x = \sin\theta$. Then $dx = \cos\theta d\theta$, and $\sqrt{1-x^2} = \cos\theta$. Substituting, we have

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta = \int \sin^2\theta\,d\theta.$$

By the double angle formula, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. Integrating, we see that

$$\int \sin^2\theta\,d\theta = \frac{1}{2}\int 1-\cos 2\theta\,d\theta = \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C.$$

Converting back to x,

$$\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) = \frac{1}{2}\left(\sin^{-1}x - x\sqrt{1-x^2}\right) + C.$$

2. Compute the volume of the solid obtained by rotating the region enclosed by $y = x$ and $y = x^2$ around the y-axis.

We are revolving around the y-axis, so we will integrate with respect to y. The outer radius is $x = \sqrt{y}$ and the inner radius is $x = y$. The bounds of integration are the points of intersection, $y = 0$ and $y = 1$. Taken together, the volume is

$$\pi\int_0^1 (\sqrt{y})^2 - y^2\,dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \pi\frac{1}{6}.$$

3. Find the sum of each of the following series.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 5 \cdot 2^{n+1}}{e^n}$

   Geometric series with ratio $r = -\frac{2}{e}$ ($|r| < 1$) and first term $a = \frac{40}{e^2}$, converges to $\frac{a}{1-r} = \frac{40}{e^2+2e}$.

   (b) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

   Telescoping, partial sum $S_n = \frac{1}{3} - \frac{1}{2n+1}$, series converges to a sum of $\frac{1}{3}$.

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 2^{2n+1}}$

   This is the Taylor series for $\sin(x)$, with $x = \frac{\pi}{2}$. So the series converges to $\sin(\frac{\pi}{2}) = 1$.

   (d) $\displaystyle\sum_{n=1}^{\infty} n \left(\frac{9}{10}\right)^{n+1}$

   Differentiate $f(x) = \frac{1}{1-x}$ to get $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$. Let $x = \frac{9}{10}$ and multiply by $(\frac{9}{10})^2$ to get a sum of 81.

4. As it happens,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

If we estimate the sum by adding from $n = 1$ to $n = N$

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n^2}$$

what should N be to *guarantee* an approximation of $\frac{\pi^2}{12}$ accurate to .0000005?

Let's use the alternating series error bound, $|S - S_N| < a_{N+1}$. So we want

$$\frac{1}{(N+1)^2} < .0000005 = 5 \cdot 10^{-7}.$$

So we want

$$5(N+1)^2 > 10^7.$$

In other words, we want

$$(N+1)^2 > 2 \cdot 10^6$$

Taking square roots, we see that we need

$$N + 1 > \sqrt{2} \cdot 10^3 > 1414.2.$$

So let's take $N = 1414$.

5. Find the mass m and center of mass $(\bar{x}, \bar{y})$ of the plate enclosed by the x-axis and $y = \ln x$ on $[1, e]$ with constant density ρ.
The mass is

$$M = \int_1^e \rho \ln x \, dx = \rho \left[x \ln x - x \right]_1^e = \rho(e - e - 0 + 1) = \rho.$$

The moment about the y-axis is

$$M_y = \int_1^e x\rho \ln x \, dx = \rho \left[\frac{1}{4}(2x^2 \ln x - x^2) \right]_1^e = \frac{\rho}{4}(e^2 + 1).$$

To find the moment about the x-axis, we need to solve for x as a function of y: $x = e^y$. The moment is then

$$M_x = \int_0^1 y\rho(e - e^y) \, dy = \frac{\rho}{2}(e - 2).$$

The centers of mass are

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{\rho}{4}(e^2 + 1)}{\rho} = \frac{e^2 + 1}{4}, \quad \bar{y} = \frac{M_x}{M} = \frac{e - 2}{2}.$$

6. Find the radius and interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{(x - 10)^n}{n 4^n}$$

Let's use the ratio test: the series converges if

$$\lim_{n \to \infty} \left| \frac{(x - 10)^{n+1}}{(n + 1)4^{n+1}} \cdot \frac{n 4^n}{(x - 10)^n} \right| < 1.$$

Evalutaing the limit, we see that

$$\lim_{n \to \infty} \left| \frac{(x - 10)^{n+1}}{(n + 1)4^{n+1}} \cdot \frac{n 4^n}{(x - 10)^n} \right| = \lim_{n \to \infty} |x - 10| \frac{n}{4(n + 1)} = \frac{|x - 10|}{4}.$$

The series therefore converges when

$$|x - 10| < 4,$$

making the radius of convergence 4. Now we check the endpoints: $x = 6$ and $x = 14$. When $x = 6$, we get the alternating harmonic series, which converges. When $x = 14$, we get the harmonic series, which diverges. Thus, the interval of convergence is

$$[6, 14).$$

7. Use any method to write down the Taylor series centered at $a = 0$ for the function $f(x) = x^2 e^{x^2}$. Use your answer to determine $f^{(8)}(0)$.

Substitute x^2 into the Taylor series for e^x, then multiply by x^2 to get $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$. Using $n = 3$, we see that the degree 8 term of this series is $\frac{1}{3!}x^8$. Recalling that the coefficient of x^8 is $\frac{f^{(8)}(0)}{8!}$, we have $\frac{f^{(8)}(0)}{8!} = \frac{1}{3!}$, so $f^{(8)} = \frac{8!}{3!}$.

8. You have succeeded John Hammond as facilitator of Jurassic Park.

Foolishly, you decided to again introduce velociraptors into the park. You clock one of the raptors as it starts at rest, accelerates to its top speed, and then immediately starts slowing down.

The raptor's speed is given by

$$v(t) = 100te^{-\frac{t}{2}},$$

where t is in seconds, and $v(t)$ is in ft/s.

How long does the raptor take to hit its top speed? What is its top speed in ft/s? In mph?

What is the raptor's average speed over the interval $[0, 15]$?

The raptor hits its top speed when $v'(t) = 0$. Differentiating tells us we want t so that

$$100\left(t \cdot \frac{-1}{2}e^{-t/2} + e^{-t/2}\right) = 0.$$

Factoring, we want

$$100e^{-t/2}\left(\frac{-t}{2} + 1\right) = 0.$$

This only happens when $t = 2$. So the raptor hits its top speed after 2 seconds. The raptor's top speed is $v(2) = \frac{200}{e}$ ft/s. This is

$$\frac{200}{e}\frac{\text{ft}}{\text{s}} \cdot \frac{\text{mi}}{5280 \text{ ft}} \cdot \frac{3600 \text{ s}}{\text{hr}} = \frac{200 \cdot 3600}{e \cdot 5280}\text{mph}.$$

To calculate the average speed of the raptor, we use the average value formula:

$$\bar{v} = \frac{1}{15 - 0}\int_0^{15} v(t)\,dt = \frac{1}{15}\int_0^{15} 100te^{-t/2}\,dt = \frac{1}{15}\left[-200e^{-t/2}(t+2)\right]_0^{15}.$$

Evaluating, we get

$$\frac{1}{15}\left[-200e^{-15/2}(17) + 400\right] \text{ ft/s}.$$