Series - summing it all up

Here's a list of all of the convergence tests for series that you know so far:

- Divergence test (a.k.a. n-th term test)
- Geometric series test
- Telescoping series
- Integral test
- p-series (including harmonic series)
- Term-size comparison test (also known as "The Comparison Test" or "Direct Comparison Test")
- Limit comparison test
- Alternating series test
- Absolute convergence implies convergence
- The Ratio test
- Remainder estimates for integral test and alternating series

Here are the details:

The Divergence test: When you're given a series $\sum_{n=1}^{\infty} a_n$, first check the limit of the underlying sequence. If $\lim\limits_{n\to\infty} a_n \neq 0$, then you can conclude that the given series $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim\limits_{n\to\infty} a_n = 0$, you can't conclude anything yet and you have to do more work.

Geometric series test: You can recognize a geometric series because it is built from an exponential sequence. An infinite geometric series generally has the form $\sum_{n=1}^{\infty} ax^{n-1}$. An infinite geometric series diverges if $|x| \geq 1$, and converges if $|x| < 1$. If an infinite geometric series converges, it converges to a sum of $\dfrac{a}{1-x}$. (Finite geometric series always converge, don't forget we have a special formula for their sums.)

Telescoping series: Telescoping series can be written in the form $\sum_{i=1}^{\infty} (a_i - a_{i+1})$. Write out the nth partial sum to see that the terms cancel in pairs, collapsing to just $a_1 - a_{n+1}$. Take the limit to see if the series converges or diverges.

The Integral test: (for positive term series only) Generally, this is our last resort, because to use it we must evaluate the corresponding improper integral, and also make sure the corresponding function is decreasing and positive.

Suppose $a_n = f(n)$, where $f(x)$ is continuous, decreasing and positive.

- If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remainder Estimate for the Integral Test: If you can use the Integral Test to show a series is convergent, then you can get a bound for how accurately the nth partial sum estimates the sum of the series: Suppose $a_n = f(n)$,where f is a continuous, positive decreasing function for $x \geq n$, and $\sum a_n$ is convergent. If $R_n = S - S_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

***p*-series:** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. If $p = 1$ we have the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

Term-size comparison test: (for positive term series only)
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms

- If $\sum b_n$ converges, and $a_n \leq b_n$ for all n beyond a certain value, then $\sum a_n$ also converges.
- If $\sum b_n$ diverges, and $a_n \geq b_n$ for all n beyond a certain value, then $\sum a_n$ also diverges.

Limit comparison test: (for positive term series only)
Suppose $a_n > 0$ and $b_n > 0$ for all n. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is finite and not zero, then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Alternating series test:
The terms in an alternating series alternate signs. They have the form $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ (if the first term is positive) or $\sum_{n=1}^{\infty}(-1)^{n}a_n$ (if the first term is negative), where $a_n > 0$. If $0 < a_{n+1} \leq a_n$ for all n (i.e., the sequence is decreasing) and $\lim\limits_{n\to\infty} a_n = 0$, then the series converges.

Remainder estimate for alternating series:
If you have an alternating series for which you can use the alternating series test to show convergence, then you can get a bound for how accurately the nth partial sum estimates the sum of the series:
Let $S_n = \sum\limits_{i=1}^{n}(-1)^{i-1}a_i$ be the nth partial sum of an alternating series and let $S = \lim\limits_{n\to\infty} S_n$ be the sum of the infinite series. Suppose that $0 < a_{n+1} \leq a_n$ for all n and that $\lim\limits_{n\to\infty} a_n = 0$. Then

$$R_n = |S - S_n| < a_{n+1}$$

Absolute convergence implies convergence:
If we have a series $\sum_{n=1}^{\infty} a_n$ that has a mix of negative and positive terms, but that doesn't necessarily alternate, sometimes it is useful to consider $\sum_{n=1}^{\infty} |a_n|$. Here's why:

$$\text{If } \sum_{n=1}^{\infty} |a_n| \text{ converges, then so does } \sum_{n=1}^{\infty} a_n.$$

We call a series *absolutely convergent* when $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, if a series is absolutely convergent, it must also be convergent. We call a series *conditionally convergent* if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.
So there are three distinct possibilities for a series: it either converges absolutely, converges conditionally, or diverges.

The Ratio test: Suppose you calculate the following limit, and

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

- If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $L > 1$ (including if $L = \infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$, we can make no conclusion about the series using this test.

The ratio test is typically useful if a series has geometric components and/or factorial components, possibly mixed with power functions. It will be extremely important in determining convergence of power series.