- Flow in: If there are V_0 mL of fluid in a tank at time $t = 0$, and fluid is flowing in at a rate of $P(t)$ mL/min, then at time t the volume of the fluid in the tank is

$$V(t) = \underline{\qquad V_0 + \int_0^t P(s)\,ds \qquad}$$

- Flow in and flow out: If there are V_0 mL of fluid in a tank at time $t = 0$, and the fluid is flowing in at a rate of $P(t)$ mL/min and dropping out of the bottom at a rate of $D(t)$ mL/min, then at time t the volume of the fluid in the tank is

$$V(t) = \underline{\qquad V_0 + \int_0^t P(s) - D(s)\,ds \qquad}$$

1. Snow is falling at a rate $r(t) = \dfrac{6t - 3t^2}{t^2 - 2t + 2}$, in inches per hour, where $t = 0$ at 8 am and t=2 at 10 am.

   (a) When is the rate of snowfall zero?

   Solution: Setting $r(t) = 0$ and solving, $\dfrac{6t - 3t^2}{t^2 - 2t + 2} = \dfrac{3t(t-2)}{t^2 - 2t + 2} = 0$, so $t = 0$ or $t = 2$. At 8 am and 10 am.

   (b) When $t = 1$, what is $r(t)$ and what does it represent? Include units.

   Solution: $r(1) = 3$. This means that snow is falling at a rate of 3 inches per hour at 9 am.

   (c) What is the maximum value of $r(t)$ and when does it occur? Interpret in the context of the problem.

   Solution: Using Wolfram Alpha, $r'(t) = \dfrac{-12(t-1)}{(t^2 - 2t + 2)^2}$. $r'(t) = 0$ at $t = 1$. $r'(t)$ is negative to the left of $t = 1$ and positive to the right of $t = 1$. By the first derivative test the relative maximum rate of snowfall is at $t = 1$. The rate at that time is 2 inches per hour. So it is snowing the hardest at 9 am.

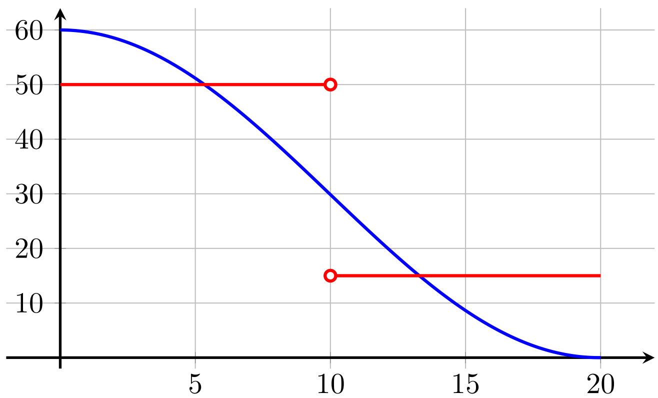
   (d) How much snow has fallen between 8 am and 10 am?

   Solution: Total amount of snow that has fallen is
   $$\int_0^2 \frac{6t - 3t^2}{t^2 - 2t + 2}\,dt = \int_0^2 \frac{6}{t^2 - 2t + 2} - 3\,dt = 6\int_0^2 \frac{1}{(t-1)^2 + 1}\,dt$$
   $$= 6\arctan(t-1)\Big|_0^2 - 6 = 6(\pi/4 + \pi/4) - 6 = 3\pi - 6 \approx 3.425 \text{ inches.}$$

2. A 1 L bottle contains 200 mL of water. Water is being poured in at a rate $P(t) = 30 + 30\cos\left(\frac{\pi t}{20}\right)$, from $t = 0$ minutes to $t = 20$ minutes. Water is simultaneously dripping out at a rate $D(t)$ defined by

$$D(t) = \begin{cases} 50 & 0 \leq t \leq 10 \\ 15 & 10 < t \leq 20 \end{cases}$$

(a) Graph $P(t)$ and $D(t)$ below



(b) How many milliliters of water were poured into the tank from $t = 0$ to $t = 20$?

Solution: The water entering the tank is given by $\displaystyle\int_0^{20} P(t)\,dt = \int_0^{20} 30+30\cos\left(\frac{\pi t}{20}\right)$. Evaluating using technology gives 600 mL.

(c) When is the total volume of water in the tank increasing? Justify your answer.

Solution: $\frac{dV}{dt} = P(t) - D(t)$, so the volume is increasing when $P(t) > D(t)$. We find the intersection points on the graph using technology, solving $30 + 30\cos\left(\frac{\pi t}{20}\right) = 50$ at $t \approx 5.354$ and $30+30\cos\left(\frac{\pi t}{20}\right) = 15$ at $t = \frac{40}{3} \approx 13.333$. So $V(t)$ is increasing on $(0, 5.354)$ and $(10, 13.333)$.

(d) What is the maximum amount of water in the bottle during the time $0 \leq t \leq 20$? At what time does this occur? Fully explain.
Solution: The absolute maximum of $V(t)$ occurs at critical values (where $\frac{dV}{dt} = 0$ or where $\frac{dV}{dt}$ does not exist) or at endpoints. $\frac{dV}{dt} = P(t) - D(t) = 0$ at the intersection points $t = 5.354$ and $t = 13.333$. $\frac{dV}{dt}$ is undefined at $t = 10$. So we calculate the volume at $t = 0$, $t = 5.354$, $t = 10$, $t = 13.333$ and $t = 20$. At $t = 0$, the volume is 200 mL. At $t = 5.354$, $V = 200 + \int_0^{5.354} 30 + 30\cos(\pi t/20) - 50\,dt = 235.264$. At $t = 10$, $V = 200 + \int_0^{10} 30 + 30\cos(\pi t/20) - 50\,dt = 190.99$. At $t = 13.333$, $V = 190.99 + \int_{10}^{13.333} 30 + 30\cos(\pi t/20) - 15\,dt = 215.403$. At $t = 20$, $V = 215.403 + \int_{13.333}^{20} 30 + 30\cos(\pi t/20) - 15\,dt = 150$. The maximum amount of water in the tank is 235.264, at $t = 10$.

3. Runners from the Bolder Boulder are finishing their 10K race at Folsom Field. at 9:00 am there are 1200 people in the stadium. Between 9:00 and 9:30 runners are arriving at the rate $P(t)$ shown in the table below, and then exiting the stadium at a constant rate of 75 people per minute. Time t is in minutes after 9:00, and the rate is given in people per minute. For answers below that require you to estimate an integral, use the Trapezoid rule.

t (in minutes)	0	10	20	30
$P(t)$	100	150	200	175

(a) Estimate the number of runners arriving at the stadium between 9:00 and 9:30.

Solution: The number of runners arriving is given by $\int_0^{30} P(t)\,dt$. Using the trapezoid rule, this is estimated by $\frac{10}{2} \cdot (100 + 2 \cdot 150 + 2 \cdot 200 + 175) = 4875$ runners.

(b) Do you think the rate at which people are arriving is increasing or decreasing at 9:25?

Solution: From the data, $P(t)$ is decreasing between 20 and 30 minutes.

(c) What is the average rate people are entering the stadium between 9:00 and 9:30? Include units.

Solution: We are asked for the average of $P(t)$. Using the average value formula, we have $P_{ave} = \frac{1}{30-0}\int_0^{30} P(t)\,dt \approx \frac{1}{30} \cdot 4875 \approx 162.6$. The average rate is therefore 162.6 runners per minute.

(d) What is the average rate of change of $P(t)$ between 9:00 and 9:30? Include units.

Solution: The average rate of change of $P(t)$ is just $\frac{P_{\text{final}} - P_{\text{initial}}}{30-0} = \frac{-75}{30} = -1.5$ people per minute per minute.

(e) Estimate the number of people in the stadium at 9:30.

Solution: Number of people in the stadium

$$= 1200 + \int_0^{30} P(t) - D(t)\,dt = 1200 + \int_0^{30} P(t) - 75\,dt \approx 1200 + 4875 - 2250 = 3825.$$