Background knowledge:
In the following statement, $f(x)$ is a function, $T_n(x)$ is its nth-degree Taylor polynomial centered at a, and the remainder $R_n(x) = f(x) - T_n(x)$.

Taylor's Inequality: If $f^{(n+1)}$ is continuous and $\left|f^{(n+1)}\right| \leq M$ between a and x, then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

1. In this first example, you know the degree n of the Taylor polynomial, and the value of x, and will find a bound for how accurately the Taylor Polynomial estimates the function.

   (a) Write down the 2nd degree Taylor Polynomial for $f(x) = e^x$ centered at $a = 0$.

   (b) If we want to use the Taylor Polynomial above to estimate e, what should x be?

   (c) Use the Taylor Polynomial from part (a) to estimate e.

   (d) Find an upper bound for $f'''(x)$ for x between a and the value at which we are estimating the function, that is, between 0 and 1. This is what we call M.

   (e) Write down the error bound for $R_2(x)$, filling in values for x, a, n and M. What does this say about the accuracy of your estimate in part c?

2. In this problem you'll know the value of x and the accuracy you're going for, and you will find how large a degree n for the Taylor Polynomial is needed.

   (a) Say that you want to estimate e to within 0.1. How many terms of the Taylor series do you need to add up? This time first find a bound M for $f^{(n+1)}(x)$ between a and x (notice you need to do this for arbitrary n). Then write down the error bound for $R_n(x)$, filling in values for x, a and M. Set this error bound to be less than 0.1 and solve for n.

   (b) Add the number of terms you found were needed to get an estimate of e to within 0.1.

3. In this problem you'll know the degree n of the Taylor Polynomial and the accuracy you're going for, and you will find out how large x can be. Using the 5th degree Taylor Series for $\sin x$ centered at $a = 0$ to estimate $\sin x$, how large can x be to get an estimate within .0005?

4. In this problem you show that a Taylor Series for a function actually converges to the function. Show that the Taylor Series for $f(x) = \sin x$ converges to $\sin x$ for all x. This background information will be useful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all } x.$$

Outline of strategy:

- Get an upper bound M for $|f^{(n+1)}(x)|$ on the interval from a to x.

- Write down the nth degree error bound for $R_n(x)$.

- Take the limit of this bound for $R_n(x)$ as $n \to \infty$, show it is 0, for all x.

- State the conclusion:

More practice:

5. (a) Find the Taylor Series directly (using the formula for Taylor Series) for $f(x) = \ln(x+1)$, centered at $a = 0$.

(b) How accurate will the estimate be if we use the fifth degree Taylor polynomial to estimate $\ln 4$? Note, you can still use Taylor's error formula for x outside the interval of convergence.

(c) Show that this series converges to $\ln(x+1)$ on the interval $(-\frac{1}{2}, \frac{1}{2})$. Note: do not use the ratio test, since it only shows convergence of the series, not convergence to the correct function. Instead, show that the limit of the error term is 0.

(d) For $x = \frac{1}{4}$, what degree Taylor polynomial do we need to use to guarantee an approximation correct to within 4 decimal places (that is, to within .00005)?

6. Show that the 6th degree Taylor Polynomial for $\cos x$, centered at 0, gives values which are accurate to at least four decimal places (to within .00005) if $|x| < 1$.

7. Find $\sin(35^\circ)$ to within 3 decimal places (to within .0005). Note that you have options here about where to place the center a.