Goal: An introduction to the idea of recursively defined sequences, meaning: sequences where each term is defined by a formula involving the previous term (or terms).

1. Sometimes sequences can be described recursively in addition to their more familiar explicit forms.

   (a) Consider the sequence defined by $a_1 = 5, \quad a_n = a_{n-1} + 3$. Write down the first 5 terms of the sequence. Identify what type of sequence it is, and find an explicit formula for a_n.

   Solution: The first few terms of the sequence are $5, 8, 11, 14, 17$. There is a common difference between the terms, so it is an arithmetic sequence. The formula will be linear, $a_n = 5 + 3(n-1) = 2 + 3n$.

   (b) Now look at the sequence defined by $a_1 = 3, \quad a_n = 2a_{n-1}$. Write down the first 5 terms of the sequence. Identify what type of sequence it is, and find an explicit formula for a_n.

   Solution: The first few terms of the sequence are $3, 6, 12, 24, 48$. There is a common ratio between the terms, so it is a geometric sequence. The formula will be linear, $a_n = 3 \cdot 2^{n-1}$.

   (c) Now look at the sequence defined by $a_1 = 5, \quad a_n = na_{n-1}$. Write down the first 5 terms of the sequence. Identify what type of sequence it is, and find an explicit formula for a_n.

   Solution: The first few terms of the sequence are $5, 10, 30, 120, 600$. To get from one term to the next, I'm multiplying by the next integer. This is like the factorial sequence, except that I started with 5 instead of 1. $a_n = 5 \cdot n!$.

2. A sequence that is increasing and bounded above must converge. A sequence that is decreasing and bounded below also must converge. Draw a picture and explain intuitively why this must be so.

   Solution: A sequence that is increasing keeps climbing, but if it is bounded above it runs out of room and eventually must settle down on something. In fact, it converges to its least upper bound. Similarly, decreasing sequences that are bounded below converge to their greatest lower bound.

3. Suppose that the sequence a_n converges and that $\lim_{n\to\infty} a_n = L$. Does the sequence a_{n+1} converge, and if so what is $\lim_{n\to\infty} a_{n+1}$? Explain.

   Solution: The sequence a_{n+1} is essentially the same sequence as a_n, it is just one step ahead. So their end behavior is the same, their limit must be the same. So $\lim_{n\to\infty} a_{n+1} = L$, too.

4. Consider the recursive sequence defined by $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}$

   (a) Write out the first 4 terms, and calculate their approximate values.

   Solution: $a_1 = \sqrt{2} \approx 1.414$
   $a_2 = \sqrt{2+\sqrt{2}} \approx 1.848$
   $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \approx 1.961$
   $a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \approx 1.990$

   (b) Every term in this sequence is bounded above by 2. For example, let's look at a_4. Explain why $a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} < \sqrt{2+\sqrt{2+\sqrt{2+2}}}$, then simplify to show $a_4 < 2$.

   Solution: Replacing the innermost $\sqrt{2}$ with a 2 in $a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$ increases its size. So $a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} < \sqrt{2+\sqrt{2+\sqrt{2+2}}}$. When we simplify, the expression collapses for us, giving
   $\sqrt{2+\sqrt{2+\sqrt{2+2}}} = \sqrt{2+\sqrt{2+\sqrt{4}}} = \sqrt{2+\sqrt{2+2}} = \sqrt{2+\sqrt{4}} = \sqrt{2+2} = \sqrt{4} = 2$. So $a_4 < 2$. This simplification works for any n, giving us $a_n < 2$. To show this rigorously, we use proof by induction, a technique taught in Discrete Mathematics.

   (c) Now show that a_n is increasing by showing that $a_{n+1} > a_n$. You'll need to use the fact that $2 > a_n$.

   $$a_{n+1} = \sqrt{2 + a_n} > \sqrt{a_n + a_n} = \sqrt{2a_n} > \sqrt{a_n \cdot a_n} = a_n$$

   (d) Explain why a_n must converge.

   Solution: We showed a_n is bounded above by 2 and a_n is increasing, so a_n must converge

(e) Now we will figure out what it converges to. Let's give a name to its limit, $\lim_{n\to\infty} a_n = L$. Take the limit of both sides of the equation below, and solve for L.

$$a_{n+1} = \sqrt{2 + a_n}$$

Solution:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n}$$
$$L = \sqrt{\lim_{n\to\infty} (2 + a_n)}$$
$$L = \sqrt{2 + L}$$
$$L^2 = 2 + L$$
$$L^2 - L - 2 = 0$$
$$(L - 2)(L + 1) = 0$$

So $L = 2$ or $L = -1$. But L must be positive. So $L = 2$.

5. Here's another recursively defined sequence $\{F_n\}$, called the sequence of *Fibonacci numbers* (which are purported to show up in nature, science, art, architecture, etc.):

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = F_1 + F_2 = 1 + 1 = 2, \quad F_4 = F_2 + F_3 = 1 + 2 = 3, \ldots,$$
$$F_n = F_{n-2} + F_{n-1}.$$

That is, the first two terms are by definition set equal to 1, and each subsequent term is the sum of the previous two.

(a) Compute and write down, F_5, F_6, F_7, F_8, F_9, and F_{10}.

Solution: 5, 8, 13, 21, 34, 55

(b) Does $\lim_{n\to\infty} F_n$ look like it exists? If so, what do you think this limit is? If not, why not?

Solution: It doesn't look like it; it looks like $\lim_{n\to\infty} F_n = \infty$.

(c) Now let's look at the sequence $\{R_n\}$ of *ratios* of successive Fibonacci numbers:

$$R_1 = \frac{F_2}{F_1} = \frac{1}{1} = 1, \quad R_2 = \frac{F_3}{F_2} = \frac{2}{1} = 2, \quad R_3 = \frac{F_4}{F_3} = \frac{3}{2} = 1.5,$$
$$R_4 = \frac{F_5}{F_4} = \frac{5}{3} = 1.666\ldots, \quad R_n = \frac{F_{n+1}}{F_n}.$$

Find the approximate value of R_5, R_6, R_7, R_8, and R_9.

Solution: 1.6, 1.625, 1.61538, 1.61905, 1.61765

(d) Does $\lim_{n\to\infty} R_n$ look like it exists? If so, what do you think this limit is? If not, why not?

Solution: It looks like the limit is about 1.62.

(e) Let's *assume* for now that $\lim_{n\to\infty} R_n$ exists and is non-zero: let's denote this limit by L. We're going to sneakily compute L. Here's how: start with the equation

$$F_n = F_{n-2} + F_{n-1}$$

defining the Fibonacci numbers. Divide both sides by F_{n-1}. This should give you an equation relating R_{n-1} and R_{n-2}. Write down a simplified version of this equation.

Solution:

$$R_{n-1} = \frac{1}{R_{n-2}} + 1.$$

(f) Now, take the limit of both sides of your above equation. What equation do you get, in terms of the limit L? (Hint: if the R_n's have a limit, then whatever they tend to as $n \to \infty$, R_{n-1} and R_{n-2} should tend to the same thing.)

Solution:

$$L = \frac{1}{L} + 1.$$

(g) Now, *solve* your above equation for L. (Some hints: (a) you may want to first do some algebra, and then apply the quadratic formula. (b) This will give you two possible solutions; why can you disregard one of them?)

Solution: The equation becomes (multiplying both sides by L):

$$L^2 = 1 + L$$

which has solution

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

But the limit must be positive (since all the terms are), so

$$L = \frac{1 + \sqrt{5}}{2} \approx 1.618.$$

(h) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_n}$ converge or diverge?

Solution: Use the ratio test. $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{F_{n+1}}}{\frac{1}{F_n}} = \lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{1}{R_n} \approx \frac{1}{1.618} \approx .618 < 1.$ So by the ratio test the series converges absolutely.