Term-size Comparison Test: necessary components for a complete solution:

- Define a_n, and make a good choice for b_n to compare to.
- Confirm that both $a_n \geq 0$ and $b_n \geq 0$.
- Determine which is larger, a_n or b_n.
- Determine the convergence/divergence of $\sum b_n$ (with a reason).
- Make a conclusion about the convergence/divergence of $\sum a_n$ (with a reason).

Example: Determine if $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-1}$ converges or diverges.

Sample of full solution:

- Let $a_n = \dfrac{n+2}{n^2-1}$. The dominant term in the numerator is n and the dominant term in the denominator is n^2. So I choose $b_n = \dfrac{n}{n^2} = \dfrac{1}{n}$.
- Both series are positive for $n \geq 2$.
- $a_n = \dfrac{n+2}{n^2-1} > \dfrac{n}{n^2-1} > \dfrac{n}{n^2} = \dfrac{1}{n} = b_n$. So $a_n > b_n$.
- $\displaystyle\sum_{n=2}^{\infty} b_n$ diverges (it is a p-series with $p = 1$).
- $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-1}$ diverges by the Term-size Comparison test.

1. Determine if $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3+n}$ converges or diverges.

Limit Comparison Test: necessary components for a complete solution:

- Define a_n, and make a good choice for b_n to compare to.
- Confirm that both $a_n \geq 0$ and $b_n \geq 0$.
- Calculate $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$. Confirm that L is positive and finite.
- Determine the convergence/divergence of $\sum b_n$ (with a reason).
- Make a conclusion about the convergence/divergence of $\sum a_n$ (with a reason).

Example: Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n-1}}$ converges or diverges.
Sample of full solution:

- Let $a_n = \dfrac{n}{\sqrt{n^5+n-1}}$. The dominant term in the numerator is n and the dominant term in the denominator is $n^{5/2}$. So I choose $b_n = \dfrac{n}{n^{5/2}} = \dfrac{1}{n^{3/2}}$.
- Both series are positive for $n \geq 1$.
- $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^5+n-1}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n \cdot n^{3/2}}{\sqrt{n^5+n-1}} = \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^5+n-1}}$
$\displaystyle= \lim_{n\to\infty} \frac{\sqrt{n^5}}{\sqrt{n^5+n-1}} = \lim_{n\to\infty} \sqrt{\frac{n^5}{n^5+n-1}} = \lim_{n\to\infty} \sqrt{\frac{n^5 \cdot \frac{1}{n^5}}{(n^5+n-1)\cdot\frac{1}{n^5}}}$
$\displaystyle= \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n^4}-\frac{1}{n^5}}} = 1$. The limit is positive and finite.
- $\sum_{n=1}^{\infty} b_n = \sum \frac{1}{n^{3/2}}$ converges (p-series, with $p = \frac{3}{2} > 1$).
- By the Limit Comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n-1}}$ also converges.

2. Determine if $\sum_{n=1}^{\infty} \frac{2n-1}{n^2+3n}$ converges or diverges.