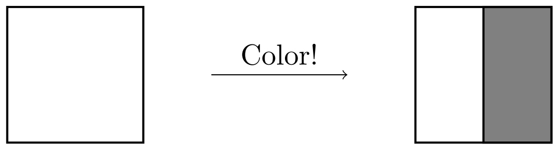
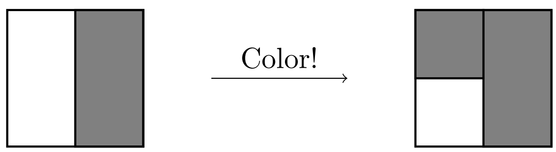
Goal: Derive the formula for the sum of a geometric series and explore the intuition behind this formula.

1. Consider coloring in a 1×1 square using the following step-by-step process. For the first step, we draw a line vertically down the middle of the square and color the right half:

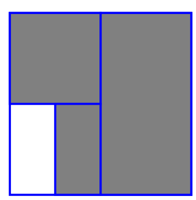


Since the square has area 1, the area of the shaded region is $1/2$. For the next step, we draw a horizontal line through the remaining space and color in the top quarter:



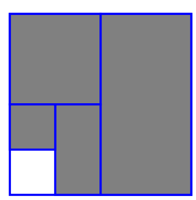
At this point, the shaded region has area $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. We continue this process, at each step coloring half of the remaining space.

(a) i. Draw the square after three steps. What is the area of the shaded region? Write this as both an expanded sum, and as a single fraction.
Solution:



Area: $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

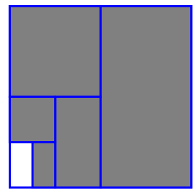
ii. Draw the square after four steps. What is the area of the shaded region? Write this as both an expanded sum, and as a single fraction.
Solution:



Area: $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$$

iii. Draw the square after five steps. What is the area of the shaded region? Write this as both an expanded sum, and as a single fraction.

Solution:



Area: $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=\frac{31}{32}.$$

(b) Now, let's think about the area after n steps, where n is an arbitrary number.

i. Write down a sum that expresses the area after n steps. Write it in both expanded form and in sigma-notation.

Solution: $$\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=\sum_{i=1}^{n}\frac{1}{2^i}.$$

ii. In Problem 1, you may have noticed a pattern in your final answers. Use this to guess a simple formula for the area after n steps (in other words, simplify so your answer is no longer a sum).

Solution: $$\frac{2^n-1}{2^n}=1-\frac{1}{2^n}.$$

iii. Based on this formula, what can you infer about the area of the shaded region as n tends toward infinity? Does this make geometric sense? Why or why not?

Solution: The area approaches 1 as n goes to infinity. This makes sense, because the square has area 1, and the shaded region is always contained inside the square. The non-shaded region gets cut in half each time, so the shaded area should approach 1.

2. Now let's talk about general geometric sequences and series. A geometric sequence is defined by an *initial term* a and *constant ratio* r, and looks like this:

$$a,\ ar,\ ar^2,\ ar^3,\ \dots.$$

Partial geometric sums are just the partial sums of geometric sequences., i.e., sums of the form

$$S_n = a + ar + \cdots + ar^{n-1} = \sum_{i=1}^{n} ar^{i-1}.$$

Infinite geometric sums, on the other hand, are the limits of the partial sums (whenever these limits exist), and look like this:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(a + ar + \cdots + ar^{n-1}\right) = \sum_{i=1}^{\infty} ar^{i-1}.$$

(a) In Problem 1, the areas that you calculated were geometric sums with a specific initial term and constant ratio. What was the initial term a in these sums? The constant ratio r?

Solution: $a = \dfrac{1}{2}, \quad r = \dfrac{1}{2}.$

In Problem 1, we came up with a simplified expression for the area of the shaded region after n steps. We can generalize this expression to arbitrary geometric sums in the following way. Let a and r be arbitrary numbers with $r \neq 1$.

(b) Using the definition $S_n = a + ar + \cdots + ar^{n-1}$, simplify $S_n - rS_n$ (Hint: Group like terms. Most of them should cancel.).

Solution:

$$\begin{aligned} S_n - rS_n &= \left(a + ar + \cdots + ar^{n-1}\right) - r\left(a + ar + \cdots + ar^{n-1}\right) \\ &= \left(a + ar + \cdots + ar^{n-1}\right) - \left(ar + ar^2 + \cdots + ar^n\right) \\ &= a - ar^n. \end{aligned}$$

(c) If you did the above calculation carefully, you should get an answer similar to

$$S_n - rS_n = a - ar^n.$$

Solve this equation to come up with a simple formula for S_n. Check that this formula agrees with your formula from Problem 1.

Solution:

$$\begin{aligned}(1-r)S_n &= a - ar^n \\ S_n &= \frac{a - ar^n}{1-r}\end{aligned}$$

Partial sum of a geometric series:

$$S_n = a + ar + \cdots + ar^{n-1} = \underline{\frac{a - ar^n}{1-r}}$$

Recall that the value of a series is defined to be the limit of its partial sums, so that

$$\lim_{n\to\infty}\left(a + ar + \cdots + ar^{n-1}\right) = \sum_{i=1}^{\infty} ar^{i-1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \underline{\frac{a - ar^n}{1-r}}.$$

(d) Based on your formula for S_n, what can you say about the convergence or divergence of S_n?

i. Does it depend on a?

Solution: No.

ii. On r?

Solution: Yes.

iii. For what values of r does the series converge?

Solution: The series converges for $|r| < 1$.

iv. For what values does it diverge?

Solution: It diverges for $|r| \geq 1$.

v. Using your formula for S_n from Problem 2(c) (on the previous page), take limits to come up with a formula for the value of the sum of a general infinite geometric series when $|r| < 1$. Check that this formula agrees with the area of the square in Problem 1.

Solution: $$\sum_{i=1}^{\infty} ar^{i-1} = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

Sum of an infinite geometric series:

$$\sum_{i=1}^{\infty} ar^{i-1} = \underline{\qquad \frac{a}{1-r} \qquad} \qquad \text{(provided } |r| < 1\text{)}$$

3. Now that we know how to calculate finite and infinite geometric sums, let's get some practice with a few examples.

(a) $\displaystyle\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^{n-1} = \frac{2}{1-(1/3)} = 3$

(b) $\displaystyle\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1/4}{1-(1/2)} = \frac{1}{2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{e}{\pi^{n-1}} = \frac{e}{1-\frac{1}{\pi}}$

(d) $\displaystyle 0.9 + 0.09 + 0.009 + \cdots = \frac{0.9}{1-0.1} = 1$

(e) $\displaystyle\sum_{n=1}^{10} 2\left(\frac{2}{3}\right)^{n-1} = \frac{2-2(2/3)^{10}}{1-(2/3)} = \frac{116,050}{19,683} \approx 5.896$

(f) $\displaystyle\sum_{n=1}^{\infty} (-2)^{n-1} =$ does not exist