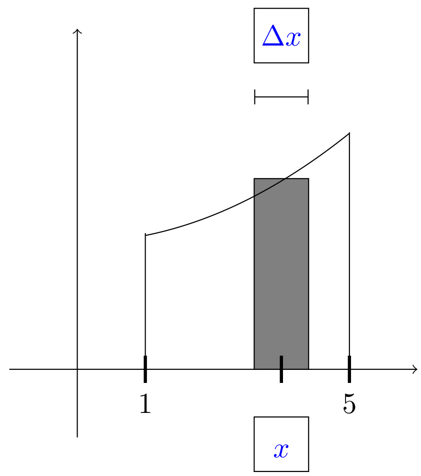
1. Area underneath a curve.

   (a) Warmup: The area of a rectangle is height × width. What is the area of a rectangle with height 7 cm and width 2 cm? Include units.

   Solution: Area = (7 cm) × (2 cm) = 14 cm^2

   (b) Consider the region between the curve $f(x) = 1 + x^2$ and the x-axis on the interval $[1, 5]$. The scale on both axes is measured in centimeters. Cut the region under the curve into thin vertical strips. The top of each strip is almost flat, so each strip is approximately a rectangle. Include units.



   i. For the rectangle with base x units from the origin, find its height and width, with units.

   height: $f(x) = 1 + x^2$ cm width: Δx cm

   ii. What is the area of this rectangle? Include units.

   Solution: Area = $(1 + x^2$ cm$) \times (\Delta x$ cm$) = (1 + x^2)\Delta x$ cm^2

   iii. To find the total area, we "add" the thin rectangles together with an integral. Express the area of the region as a definite integral. Include units.

   Solution: $\int_1^5 1 + x^2 \, dx$ cm^2

   iv. Compute the integral you found to determine the area. Include units.

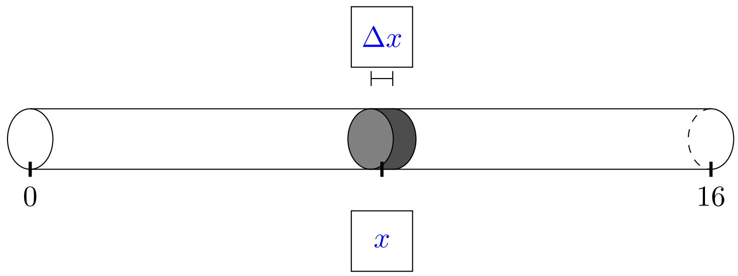
   Solution: $\int_1^5 1 + x^2 \, dx = x + \frac{1}{3}x^3 \Big|_1^5 = \frac{136}{3}$ cm^2

2. Mass of a rod.

   (a) Warmup: The mass of a thin rod is linear-density $\times$ length. Lead has volume-density of 11.34 g/cm^3. So a lead rod with diameter 1 cm would have linear-density of $\rho = 11.34$ g/cm$^3 \cdot \pi(.5\text{cm})^2 \approx 9$ g/cm. Find the mass of a 8 cm long lead thin rod that has linear-density 9 g/cm. Include units.

   Solution: Mass = (9 g/cm) $\times$ (8 cm) = 72 g

   (b) Now, suppose a rod with length 16 cm has a linear mass-density $\rho(x) = 1 + \sqrt{x}$ g/cm that depends on the distance x from one end. Cut the rod into thin slices. When cut thin enough, each slice will have approximately constant density. Fill in the boxes in the diagram below.



   i. For the slice x units from the left, find its density and length, with units.

   density: $\rho(x) = 1 + \sqrt{x}$ g/cm length (thickness): Δx cm

   ii. Find the mass of this thin slice of the rod, then express the mass of the rod as a definite integral, then evaluate the integral to determine the mass. Include units at each step.

   Solution: Mass of slice $= (1 + \sqrt{x}$ g/cm$) \times (\Delta x$ cm$) = (1 + \sqrt{x})dx$ g

   Mass of rod $= \displaystyle\int_0^{16} 1 + \sqrt{x}\,\Delta x$ g

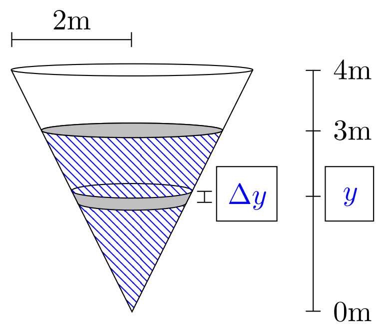
   $$= \int_0^{16} 1 + \sqrt{x}\,dx \text{ g} = x + \frac{2}{3}x^{\frac{3}{2}}\Big|_0^{16} \text{ g} = 16 + \frac{128}{3} \text{ g} = \frac{176}{3}\text{g}$$

3. Work done to pump water from a tank.

   (a) Background formula: Write down the physics formula for calculating the work done to lift an object. Include SI units.

   Solution: Work = Force × Distance, where work is measured in Joules ($J = N \cdot m$), Force is measured in (N) and distance is measured in meters (m).

   (b) Suppose you want to find the work done to pump water out of the top of the conical tank below.



   i. How far does the slice at height y travel to reach the top of the tank? Include units.

   Solution: $(4 - y)$ m

   ii. What is the weight of this slice (i.e. the gravitational force)? Include units.

   Solution: Force $= \underbrace{1000\ \text{kg/m}^3 \cdot \text{Volume}}_{\text{mass}} \cdot \underbrace{9.8\ \text{m/s}^2}_{\text{accel.}}$. The slice at height y has radius $y/2$, so its volume is $\pi(y/2)^2\, dy$ m^3. Putting everything together yields: $F = 9800\pi \left(\frac{y}{2}\right)^2 dy$ N.

   iii. Write down an expression in terms of y for the work done on this slice.

$$W_{\text{slice}} = \underbrace{9800\pi \left(\frac{y}{2}\right)^2 dy\ \text{N}}_{\text{force}} \cdot \underbrace{(4 - y)\ \text{m}}_{\text{distance}}$$

   iv. How much work is required to empty the tank of water? (Use an integral to add up the small slices of work.)
   Solution:

$$\begin{aligned}
W_{\text{total}} &= \int_0^3 W_{\text{slice}} = \int_0^3 1000\pi \left(\frac{y}{2}\right)^2 \cdot 9.8 \cdot (4 - y)\, dy \\
&= \frac{9800\pi}{4} \int_0^3 y^2(4 - y)\, dy = \frac{9800\pi}{4} \int_0^3 4y^2 - 4y^3\, dy \\
&= \frac{9800\pi}{4} \left[\frac{4}{3}y^3 - \frac{1}{4}y^4\right]_0^3 = \frac{9800\pi}{4} \left[\left(36 - \frac{81}{4}\right) - 0\right] \\
&= \frac{77175\pi}{2}\ \text{J}
\end{aligned}$$

4. Work done to launch a satellite.

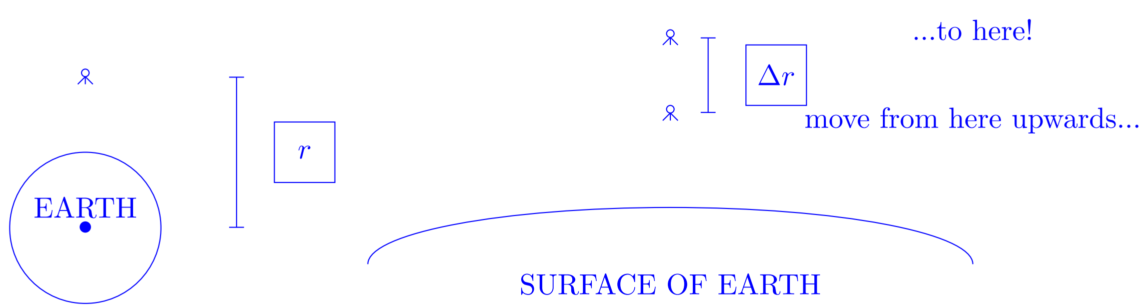
   - Context: Suppose you want to take a 1000 kg satellite from the surface of the Earth and remove it entirely from Earth's gravitational field. The force between two bodies with masses m_1 and m_2 that are a distance r apart is given by the formula

     $$F = G\frac{m_1 m_2}{r^2}$$

     (G is a constant, m_1 and m_2 are the masses, r is the distance between their centers.)

   - Following the same procedures and thought processes as in the previous examples, first approximate the work necessary to move the satellite a short distance further from earth. Then express the work done to remove the satellite from Earth's gravitational field using an improper integral. Call the radius of the earth r_e. Compute the improper integral to find the work done. Wait until the end to substitute the values $G \approx 6.67 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}$; $m_e \approx 5.97 \times 10^{24}$ kg; $r_e \approx 6370$ km.

     Solution:



Call m_1 be the mass of earth, m_2 the mass of the satellite, and r_e the radius of earth. The work to move the satellite a small distance dr is given by W $= G\frac{m_1 m_2}{r^2} \times \Delta r$ J
Total work in removing the satellite from earth's gravitational field is given by $\displaystyle\int_{r_e}^{\infty} \frac{Gm_1m_2}{r^2}\, dr$ J

$$= \int_{r_e}^{\infty} \frac{Gm_1m_2}{r^2}\, dr = \lim_{b\to\infty} -\frac{Gm_1m_2}{r}\bigg|_{r_e}^{b} = Gm_1m_2 \lim_{b\to\infty}\left(\frac{1}{r_e} - \frac{1}{b}\right) = \frac{Gm_1m_2}{r_e}$$

Converting units, $r_e = 6370000$ meters. Substituting numbers gives $\frac{Gm_1m_2}{r_e} = 6.25 \times 10^{10}$ Joules