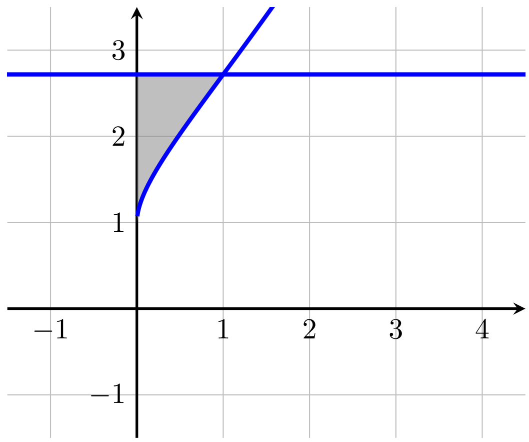
1. (a) Graph the region bounded by $y = e^{\sqrt{x}}$, $y = e$ and the y-axis.



(b) Write an integral that will give the area of this region by slicing vertically.

Solution: $\displaystyle\int_0^1 e - e^{\sqrt{x}}\, dx$

(c) Use technology to evaluate the integral. (At home try evaluating this by hand - it's a good review of techniques of integration!)

Solution: $\displaystyle\int_0^1 e - e^{\sqrt{x}}\, dx = e - 2 \approx 0.718$

(d) Write an integral that will give the area of this region by slicing horizontally.

Solution: First solve the function on the right-hand side of the region for x, $x = (\ln y^2)$. The area is given by $\displaystyle\int_1^e (\ln y)^2\, dy$.

(e) Use technology to evaluate the integral. (It's another great review problem to evaluate this by hand.)

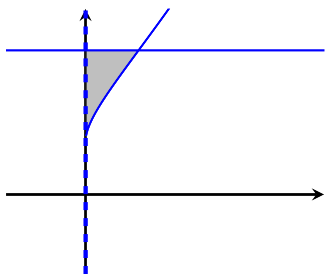
Solution: $\displaystyle\int_1^e (\ln y)^2\, dy = e - 2 \approx 0.718$

(f) Do a "sanity-check" of your numerical answers. Do they roughly match the area of the region you graphed?

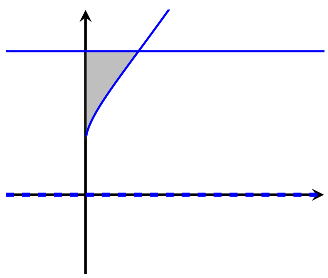
Solution: Whew, the two integrals produced the same result! Looking at my region, it is contained in a triangle whose width is 1 and whose height is $e - 1$. So my area should be somewhat less than $\frac{1}{2} \cdot 1.7 \approx 0.85$. Yup, I'm in the right ballpark!

2. Write (but do not solve) the integrals that give the volume of the solid obtained by rotating the region from the previous problem about each of the given axes.

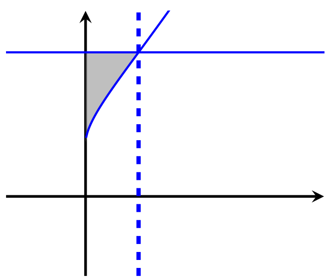
   (a) Rotate about the y-axis: **Solution:** Volume $= \pi \int_1^e ((\ln y)^2)^2 \, dy$

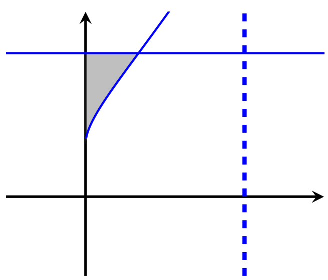
   (b) Rotate about the x-axis: **Solution:** Volume $= \pi \int_0^1 e^2 - (e^{\sqrt{x}})^2 \, dx$

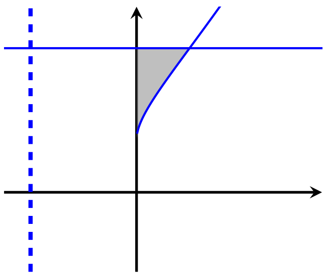
   (c) Rotate about the line $x = 1$: **Solution:** Volume $= \pi \int_1^e 1^2 - (1 - (\ln y)^2)^2 \, dy$

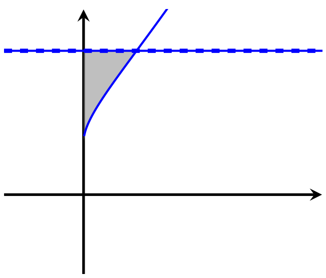
   (d) Rotate about the line $x = 3$: **Solution:** Volume $= \pi \int_1^e 3^2 - (3 - (\ln y)^2)^2 \, dy$

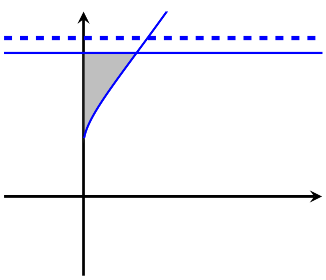
(e) Rotate about the line $x = -2$: **Solution:** Volume $= \pi \int_1^e ((\ln y)^2 + 2)^2 - 2^2 \, dy$



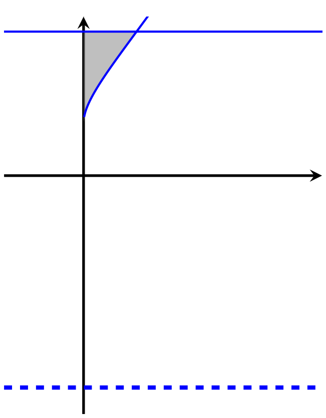
(f) Rotate about the line $y = e$: **Solution:** Volume $= \pi \int_0^1 (e - e^{\sqrt{x}})^2 \, dx$



(g) Rotate about the line $y = 3$: **Solution:** Volume $= \pi \int_0^1 (3 - e^{\sqrt{x}})^2 - (3 - e)^2 \, dx$



(h) Rotate about the line $y = -4$: **Solution:** Volume $= \pi \int_0^1 (e + 4)^2 - (e^{\sqrt{x}} + 4)^2 \, dx$



3. Using the disk-method, find the volume of a sphere of radius R.
Solution:

$$\begin{aligned}
\text{Volume} &= \pi \int_{-R}^{R} x^2\, dy \\
&= \pi \int_{-R}^{R} (R^2 - y^2)\, dy \quad \text{since } x^2 = R^2 - y^2 \\
&= \pi R^2 y - \pi \frac{y^3}{3}\Big|_{y=-R}^{y=R} \\
&= \left(\pi R^2(R) - \pi\frac{(R)^3}{3}\right) - \left(\pi R^2(-R) + \pi\frac{(-R)^3}{3}\right) \\
&= 2\pi R^3 - \frac{2}{3}\pi R^3 = \frac{4\pi R^3}{3}
\end{aligned}$$

4. Now consider the same sphere of radius R. Imagine drilling a hole through it from the north pole straight to the south pole. Since parts of the top and bottom of the sphere are removed, after drilling the hole the object is shorter than it was - the fatter the hole, the shorter the object that remains. A hole is drilled with the right width so that the object that remains has a height $2h$. To help you imagine this, if h is close to the original radius R, then the hole is narrow, and the solid looks like a spherical bead with a hole for a string to pass through. But if h is small, then what remains looks more like a ring to wear on your finger, with slightly curved sides. Find the volume of this object.
Solution: We'll use vertically stacked discs. The horizontal radius of the hole, l, can be recovered using the Pythagorean theorem, $l^2 = R^2 - h^2$. Then the area of each disc is $\pi x^2 - \pi(R^2 - h^2)$.

$$\begin{aligned}
\text{Volume} &= \pi \int_{-h}^{h} x^2 - (R^2 - h^2)\, dy = \pi \int_{-h}^{h} (R^2 - y^2 - R^2 + h^2)\, dy \quad \text{since } x^2 = R^2 - y^2 \\
&= \pi \int_{-h}^{h} (h^2 - y^2)\, dy \\
&= \pi h^2 y - \pi \frac{y^3}{3}\Big|_{y=-h}^{y=h} \\
&= \left(\pi h^2(h) - \pi\frac{(h)^3}{3}\right) - \left(\pi h^2(-h) - \pi\frac{(-h)^3}{3}\right) \\
&= \frac{4\pi h^3}{3}
\end{aligned}$$

5. Your answer should surprise you, because the value of R (the radius of the sphere) does not show up in the answer. How can this be? Interpret what this result means.
Solution: As the radius of the sphere shrinks, the diameter of the hole must also shrink so that h can remain the same. This increases the shape's volume because less volume is drilled out, but it also decreases the sphere's radius, which decreases the shape's volume. The two effects exactly cancel each other out.