The goal of this project is to develop "function sense" about the decay rate of functions. This skill is important for determining convergence of improper integrals, and it will become important again when we study convergence of series.

Problems 1-4 will help develop your *numerical* "function sense."

1. Consider the improper integral $\displaystyle\int_3^\infty \frac{1}{x^2\ln(x)}dx$. Use technology to find:

   (a) $\displaystyle\int_3^{10} \frac{1}{x^2\ln(x)}dx =$ **Solution:** 0.154022

   (b) $\displaystyle\int_3^{100} \frac{1}{x^2\ln(x)}dx =$ **Solution:** 0.184582

   (c) $\displaystyle\int_3^{1000} \frac{1}{x^2\ln(x)}dx =$ **Solution:** 0.186283

2. What can you conclude about the convergence/divergence of the improper integral $\int_3^\infty \frac{1}{x^2\ln(x)}\,dx$ based on your data from problem 1?

   (a) Based on numerical evidence, the above integral converges.
   (b) Based on numerical evidence, the above integral diverges.
   (c) Based on numerical evidence, it appears that the above integral converges.
   (d) Based on numerical evidence, it appears that the above integral diverges.

   Solution: (c)

3. Now, consider the improper integral $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}dx$. Again, use technology to find

   (a) $\displaystyle\int_3^{10} \frac{\ln(x)}{\sqrt{x}}dx =$ **Solution:** 5.03621

   (b) $\displaystyle\int_3^{100} \frac{\ln(x)}{\sqrt{x}}dx =$ **Solution:** 55.2259

   (c) $\displaystyle\int_3^{1000} \frac{\ln(x)}{\sqrt{x}}dx =$ **Solution:** 313.516

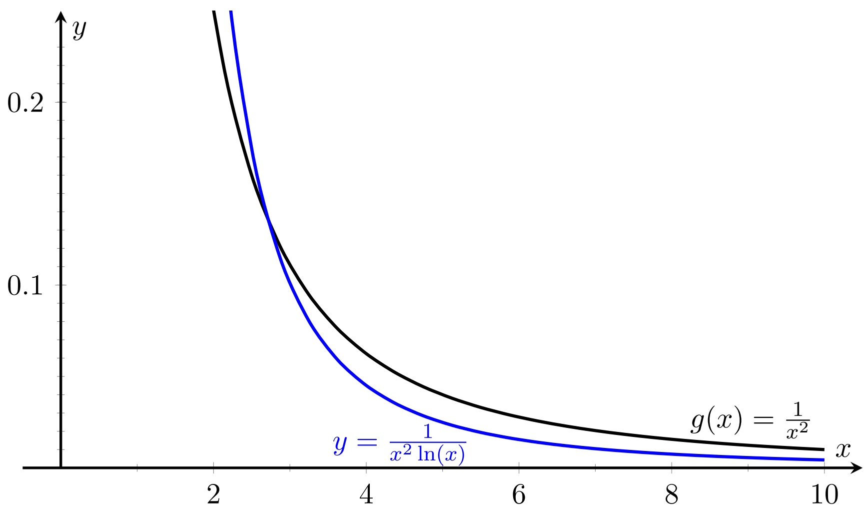
4. What can you conclude about the convergence/divergence of the improper integral $\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$ based on your data from problem 3?

   (a) Based on numerical evidence, the above integral converges.
   (b) Based on numerical evidence, the above integral diverges.
   (c) Based on numerical evidence, it appears that the above integral converges.
   (d) Based on numerical evidence, it appears that the above integral diverges.

   Solution: (d)

Problems 5-8 will help develop your *graphical* "function sense."

5. Sketch the graph of $f(x) = \dfrac{1}{x^2 \ln(x)}$. The graph of $g(x) = \dfrac{1}{x^2}$ has been provided for reference. In your sketch, identify any intersection points and pay attention to which function has higher values.



6. Using the graph you just made:

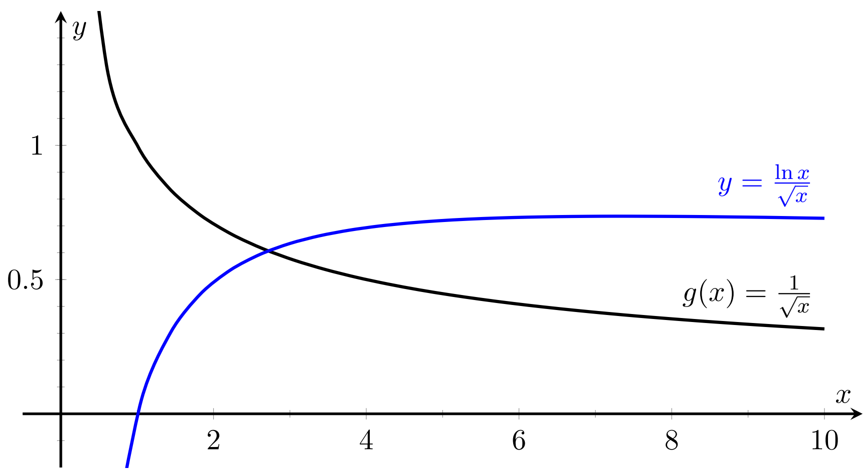
   (a) Do you think $\displaystyle\int_3^\infty \frac{1}{x^2 \ln(x)}\,dx \quad \leq \quad \int_3^\infty \frac{1}{x^2}\,dx$ is true? Why or why not?

   Solution: Yes. The graphs cross between $x = 2$ and $x = 3$, and on $[3, \infty)$ (or as close to infinity as I can see), the graph of $\frac{1}{x^2 \ln(x)}$ is squeezed between the x-axis and the graph of the $\frac{1}{x^2}$. So the area under it to the right of $x = 3$ must be smaller. More analytically, to the right of $x = 3$, $\frac{1}{x^2 \ln(x)}$ has a bigger denominator than $\frac{1}{x^2}$, so it is indeed smaller on the interval of integration.

   (b) What does this suggest about the convergence/divergence of the two integrals and why?

   Solution: Since the integral on the right converges (p¿1), it follows that the integral on the left must also converge. This is because if the right integral is finite, and the left integral is less than the right integral, the left integral must also be finite.

7. Sketch the graph of $f(x) = \dfrac{\ln(x)}{\sqrt{x}}$. The graph of $g(x) = \dfrac{1}{\sqrt{x}}$ has been provided for reference. In your sketch, identify any intersection points, and pay attention to which function is higher. Do these functions have aysmptotes?



8. Using the graph you just made

   (a) Do you think $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx \quad \geq \quad \int_3^\infty \frac{1}{\sqrt{x}}\,dx$ is true? Why or why not?

   Solution: Yes, because, on $[3, \infty)$, or as far towards infinity as I can see, the graph of the integrand on the left lies above the graph of the integrand on the right.

   (b) What does this suggest about the convergence/divergence of the two integrals and why?

   Solution: Since the integral on the right diverges ($p < 1$), the integral on the left must also diverge. This is because if the right integral is not finite, and the left integral is greater than the right integral, it also cannot be finite.

9. What advice would you give to a classmate who says "in Problem 8, we found that the function $f(x) = \frac{\ln(x)}{\sqrt{x}}$ diverges"?

   Solution: Improper integrals are limits of areas, so it makes sense to say that they converge or diverge. Functions are not limits. They do not converge or diverge. In Problem 8, we found that the *improper integral* $\displaystyle\int_3^\infty \frac{\ln(x)}{\sqrt{x}}\,dx$ diverges.

10. The punchline:

> **Comparison Theorem for Integrals**
> If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then
>
> (a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.
>
> (b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

If we know $\int_a^\infty f(x)dx$ diverges, what can we conclude about $\int_a^\infty g(x)dx$? **Solution:** Nothing.

And if we know $\int_a^\infty g(x)dx$ converges, what can we conclude about $\int_a^\infty f(x)dx$? **Solution:** Nothing.

Putting it into practice:

11. Consider the integral $\displaystyle\int_2^\infty \frac{\cos^2(x)}{x^2}dx$.

    (a) Do you think this improper integral converges or diverges?
    Solution: Converges

    (b) A good comparison function is:
    Solution: $1/x^2$

    (c) Write down the inequality that compares $\dfrac{\cos^2(x)}{x^2}$ to your answer to (b).
    Solution: $\dfrac{\cos^2(x)}{x^2} \leq \dfrac{1}{x^2}$

    (d) How will this inequality help you prove that your guess is correct?
    Solution: If the integral on the right converges, then the integral on the left must also converge.

12. Consider the integral $\displaystyle\int_1^\infty \frac{1+\sin^4(2x)}{\sqrt{x}}dx$.

    (a) Do you think this improper integral converges or diverges?
    Solution: Diverges

(b) A good comparison function is:

Solution: $1/\sqrt{x}$

(c) Write down the inequality that compares $\dfrac{1+\sin^4(2x)}{\sqrt{x}}$ to your answer to (b).

Solution: $\dfrac{1+\sin^4(2x)}{\sqrt{x}} \geq \dfrac{1}{\sqrt{x}}$

(d) How will this inequality help you prove that your guess is correct?

Solution: If the integral on the right diverges, then the integral on the left must also diverge.

13. Consider the integral $\displaystyle\int_2^\infty \frac{1}{x+e^x}dx$.

(a) Do you think this improper integral converges or diverges?

Solution: Converges

(b) A good comparison function is:

Solution: $1/e^x$

(c) Write down the inequality that compares $\dfrac{1}{x+e^x}$ to your answer to (b).

Solution: $\dfrac{1}{x+e^x} \leq \dfrac{1}{e^x}$

(d) How will this inequality help you prove that your guess is correct?

Solution: If the integral on the right converges, then the integral on the left must also converge.

14. Consider the integral $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-1}}dx$.

(a) Do you think this improper integral converges or diverges?

Solution: Diverges

(b) A good comparison function is:

Solution: $1/x$

(c) Write down the inequality that compares $\dfrac{1}{\sqrt{x^2-1}}$ to your answer to (b).

Solution: $\dfrac{1}{\sqrt{x^2-1}} \geq \dfrac{1}{x}$

(d) How will this inequality help you prove that your guess is correct?

Solution: If the integral on the right diverges, then the integral on the left must also diverge.

The limits and limitations of your "function sense."

15. Consider the integral $\int_1^\infty \frac{1}{\sqrt{x^2+1}}dx$

    (a) Do you think this improper integral converges or diverges? **Solution:** Diverges

    (b) A good comparison function is:

    Solution: $\frac{1}{x}$

    (c) Write down an inequality that compares $\frac{1}{\sqrt{x^2+1}}$ to your answer to (b).

    Solution: $\frac{1}{\sqrt{x^2+1}} \leq \frac{1}{x}$

    (d) How will this inequality help you prove your guess? (or will it?)

    Solution: It won't help. Knowing the integral is smaller than a divergent integral tells us nothing.

16. If your attempt at comparison above failed, then make a guess about the convergence/divergence of this integral by investigating the above integral numerically.

    Solution: Numerical investigation provides evidence that the integral is divergent.

17. Even though $\frac{1}{x}$ did not produce the desired inequality, it may still provide a useful comparison.

    (a) Show that $\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2+1}}}{\frac{1}{x}} = 1$

    Solution: $\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^2+1}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2+1}} = \sqrt{\lim_{x\to\infty} \frac{x^2}{x^2+1}} = \sqrt{1} = 1$

    (b) What does this say about the relationship between $\frac{1}{\sqrt{x^2+1}}$ and $\frac{1}{x}$?

    Solution: As x approaches infinity, the functions approach the same value.

18. Try to write your own theorem that involves the limit idea above. I will get you started.

The Limit Comparison Test: If $f(x)$ and $g(x)$ are positive continuous functions, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} =$ __________, then:

Solution: If $f(x)$ and $g(x)$ are positive continuous functions, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$, where $0 < c < \infty$, then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge or diverge.