This worksheet will analyze the spread of Ebola through interaction between infected and susceptible people. Ebola is an infectious and extremely lethal viral disease that first surfaced in humans in the 1970s in Central Africa. Because it kills approximately 70% of the people who contract it, it can wipe out entire populations when it strikes. The World Health Organization (WHO) models the spread of this disease with differential equations to predict what interventions will work to stop the spread of the disease.

In this model, the population is broken into three groups: $S(t)$, the number of susceptible people remaining, $I(t)$, the number of infected people who can infect others, and $R(t)$, the number removed through quarantine, death, or recovery. In this example, time (t) is the number of weeks since the epidemic was discovered in an isolated town with a population of 10,000 people. On the first day, the town had 20 cases of Ebola.

1. What is $S(t) + I(t) + R(t)$? Does this depend on t?

   Solution: $S(t) + I(t) + R(t) = 10000$. This is the original population of the town, since every person is exactly one of susceptible, infected, or removed. This does not depend on t.

2. Our goal in this problem is to develop the differential equations necessary to make predictions about the effects of the disease and intervention on the population.

   (a) Since infections involve contact between susceptibles and infecteds, the WHO believes that the number of susceptibles will decrease over time at a rate proportional to the product of the number of susceptibles and the number of infecteds, with a constant of proportionality $-a$, where $a > 0$. Write down a formula for $\dfrac{dS}{dt}$.

   Solution:
   $$\frac{dS}{dt} = -aS(t)I(t)$$

   (b) The WHO also believes people are "removed" at a rate proportional to the number of people infected, with a constant of proportionality b. Write a formula for $\frac{dR}{dt}$.

   Solution:
   $$\frac{dR}{dt} = bI(t)$$

   (c) As in problem 1, assume the total number of people being counted is always the original population of the town. Differentiating your equation from question 1 gives $\dfrac{dS}{dt} + \dfrac{dI}{dt} + \dfrac{dR}{dt} = \underline{\ 0\ }$. Use this to write down a formula for $\dfrac{dI}{dt}$.

   Solution:
   $$\frac{dI}{dt} = aS(t)I(t) - bI(t)$$

Stop. Check your differential equations with your instructor before proceeding.

3. We've studied how to solve differential equations, but we can also use sequences to approximate the solutions, as in Euler's method. Let $S_n = S(n)$, $I_n = I(n)$, and $R_n = R(n)$, that is, we're using the subscript of the sequence to denote the number of weeks that have passed.

(a) We can approximate $\dfrac{dS}{dt}$ by setting the expression we found for it in 2(a) equal to $\dfrac{\Delta S}{\Delta t}$.

Use this approximation, along with $\Delta S = S_{n+1} - S_n$, and a step-size of $\Delta t = 1$ week, to find a recursive formula for S_{n+1} in terms of S_n and I_n.

Solution: $S_{n+1} = S_n - aS_nI_n$

(b) Use $\dfrac{dI}{dt} \approx \dfrac{\Delta I}{\Delta t}$, $\Delta I = I_{n+1} - I_n$, and a step-size of $\Delta t = 1$ week, to find a recursive formula for I_{n+1} in terms of S_n and I_n.

Solution: $I_{n+1} = I_n + aS_nI_n - bI_n$

(c) Use $\dfrac{dR}{dt} \approx \dfrac{\Delta R}{\Delta t}$, $\Delta R = R_{n+1} - R_n$, and a step-size of $\Delta t = 1$ week, to find a recursive formula for R_{n+1} in terms of R_n and I_n.

Solution: $R_{n+1} = R_n + bI_n$

4. The WHO estimates that if no interventions are put into place, then $a = 0.0005$ and $b = 0.12$. Use your answer from the previous part and a calculator to fill in the following table showing the progress of the disease through the population. Start with the first row and use 3(a), 3(b), and 3(c) to determine the next row. Continue this process until the table is filled in.

n	S_n	I_n	R_n
0	9980	20	0
1	9880	117	3
2	9302	681	17
3	6135	3767	98
4	-5420	14870	550

5. Using your data from problem 4, how long until Ebola has infected or removed everyone in the town?

   Solution: Sometime between 3 and 4 weeks.

6. The value for b can be thought of as the probability an infected person is removed each week through death, recovery, *or quarantine*. Recall that the WHO found an infected person's chance of being removed through death or recovery to be 0.12. Find the new value of b if the WHO implements a quarantine which removes 50% of those infected each week. (Hint: this should be the probability an infected person is removed through death or recovery plus the probability an infected person is removed by quarantine AND is not removed through death or recovery).

   Solution: .12 is probability an infected person is removed through death or recovery.

   .88 is then the probability an infected is not removed through death or recovery and 0.5 is the probability an infected person is removed by quarantine. Therefore $.5 \times .88$ is the probability an infected person is removed by quarantine AND is not removed through death or recovery.

   So now $b = .12 + .5 \times .88 = .56$

7. We'll recalculate, this time assuming that quarantining began at week 1. So, using the new value for b that you just found and the data for $n = 1$ that you found in problem 6, fill in the following table.

n	S_n	I_n	R_n
1	9880	117	3
2	9302	629	69
3	6377	3202	421

8. The WHO used computer software to generate graphs of the behavior of the functions $S(t)$, $I(t)$, $R(t)$ after the quarantine was imposed rather than the recursive approximations that you found in problem 7. Note $I(t)$ includes infectious people who are not quarantined. The graphs are as follows, with units of weeks on the horizontal axis.
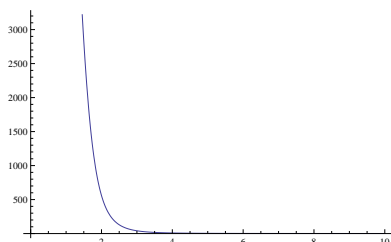
Figure 1: $S(t)$
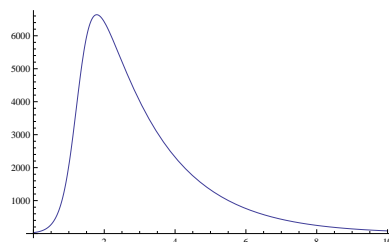
Figure 2: $I(t)$
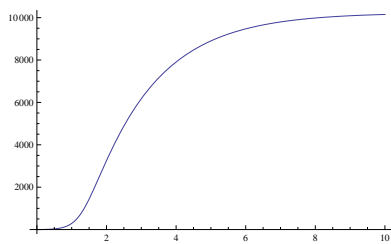
Figure 3: $R(t)$

(a) Based on these graphs, approximate the maximum of $I(t)$, that is the number of people who were infectious but not quarantined. Compare it to your approximation in problem 7. Did you get the same result? If not, why not? Do your solutions match the graph?

Solution: We see that $I(t)$ peaks at about 6700 people, when $t \approx 2$. (Note that this is very different from what we saw in the previous problem because of inaccuracy in the approximations $\frac{\mathrm{d}S}{\mathrm{d}t} \approx S_{n+1} - S_n$, $\frac{\mathrm{d}I}{\mathrm{d}t} \approx I_{n+1} - I_n$, and $\frac{\mathrm{d}R}{\mathrm{d}t} \approx R_{n+1} - R_n$. You could get a more accurate approximation if, for example, you defined S_n, I_n, and R_n in terms of days instead of weeks.)

(b) What can you say about the results of the quarantine?

Solution: Since the susceptible population approaches 0, it appears that although the quarantine may have slowed the process down, in the end the effort was ineffective because everyone was infected.