1. Warm-up (Review of related rates from Calc 1): A cylindrical keg with radius 10 cm and height 25 cm has a drain hole at the bottom with an area of 2 cm^2. The keg is draining at a constant rate of $200\dfrac{\text{cm}^3}{\text{sec}}$.

   (a) Find the depth of the liquid when there are 4 L of the liquid remaining. Use the fact that 1 L = 1000 cm^3. Do you need to use calculus for this part?

   Solution: The volume of the liquid is $\pi r^2 h$, where $r = 10$ cm, and h is what we are trying to determine. $\pi r^2 h = 4000\ \text{cm}^3$.
   $\pi(100h) = \frac{4000}{100\pi} \approx 12.73$ cm. I did not use calculus - just the formula for the volume of a cylinder.

   (b) How fast is the height of the liquid dropping where there are 4 L remaining in the keg? Do you need calculus for this part?

   Solution: $V = \pi r^2 h = 100\pi h$. Differentiate with respect to time t:

   $$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

   Solve for $\frac{dh}{dt}$:

   $$\frac{dh}{dt} = \frac{dV}{dt}\frac{1}{100\pi} = -200 \cdot \frac{1}{100\pi} \approx -.64\frac{\text{cm}}{\text{sec}}$$

   I did use calculus, but I might have been able to do it without calculus, since the flow out of the tank is constant, and the size of the cross-sections of the keg are also constant.

   (c) How long does it take for the tank to drain? Do you need calculus for this part?

   Solution: The volume of the full tank is $\pi r^2 h = \pi 10^2 \cdot 25 \approx 7654\ \text{cm}^3$. At a drain rate of $200\dfrac{\text{cm}^3}{\text{sec}}$, it takes 39.3 sec. Although I could have used calculus to solve this, I didn't need to because the drain rate is constant.

2. According to the model of the previous problem, is the liquid coming out at a constant rate? Does this model fit with your experience?

   Solution: It is constant, but in my experience tanks actually drain slower as they empty. So the model does not fit my experience.

3. Realistically, we expect the rate at which the liquid drains to depend on ___the area of the hole___, ___the depth of the liquid___, and the viscosity of the liquid.

   We will soon learn to take these factors into consideration using differential equations.

4. According to Toricelli's Law, water drains from a tank according to the following law:

$$\frac{dV}{dt} = -0.6A\sqrt{2gh(t)}$$

where A is the area of the hole, g is acceleration due to gravity, and $h(t)$ is the depth of the liquid. (The constant 0.6 is based on the viscosity of water).

(a) For the keg in problem 1, how fast is the height dropping when there are 4 L of beverage remaining?

Solution: For a cylinder, $V = \pi r^2 h$. Here $r = 10$ cm is constant, so $V = 100\pi h$ cm^2. Differentiating both sides with respect to time t yields $\frac{dV}{dt} = 100\pi \frac{dh}{dt}$. From the previous problem we know that $h \approx 12.72$ cm. Substituting gives

$$\frac{dV}{dt} = -0.6 \cdot 2 \text{ cm}^2 \sqrt{2 \cdot 980 \frac{\text{cm}}{\text{sec}^2} \cdot 12.73 \text{ cm}} = 100\pi \frac{dh}{dt}$$

Solving gives $\frac{dh}{dt} = \frac{-189.5\text{cm}}{100\pi} \approx -.60 \frac{\text{cm}}{\text{sec}}$

(b) Note that $\frac{dV}{dt}$ depends on time, as does $\frac{dh}{dt}$. Show that h satisfies the differential equation $\frac{dh}{dt} = -.1691\sqrt{h}$.

Solution:

$$\frac{dV}{dt} = -0.6A\sqrt{2gh} = 100\pi \frac{dh}{dt}$$

Substituting $A = 2$ cm^2, $g = 980 \frac{\text{cm}}{\text{sec}^2}$ gives

$$\frac{dh}{dt} = \frac{-1.2\sqrt{1960}}{100\pi} \sqrt{h} \frac{\text{cm}}{\text{sec}} = -.1691\sqrt{h}$$

(c) Solve the differential equation to find h. Use the initial condition that the $h = 25$ cm when $t = 0$ to solve for the constant.

Solution: Separating variables gives $h^{-1/2} dh = -.1691 dt$.
Integrating gives $2h^{1/2} = -.1691t + C$. Now $h(0) = 25$. Substituting $h = 25$ and $t = 0$ and solving gives $C = 10$. This yields the differential equation $2h^{1/2} = -.1691t + 10$. Solving for h gives $h = (-.0846t + 5)^2$.

(d) Show it takes about 59.1 seconds for the keg to drain.

Solution: Setting $h = 0$ and solving for t gives $t = 59.1$ seconds.

5. Warm-up (Review of related rates from Calc 1): A conical funnel has radius 4 cm and height 8 cm. Coffee is draining from the funnel at a constant rate of 1.3 cm^3/sec.

   (a) What is the depth of the coffee when there are 100 cm^3 of coffee remaining in the funnel? Do you need calculus for this part?

   Solution: Letting h represent the depth of the coffee and r the radius at the height of the coffee, by similar triangles we have that $\frac{4}{8} = \frac{r}{h}$, so $r = \frac{h}{2}$. The volume of a cone is $\frac{1}{3}\pi r^2 h = 100\ \text{cm}^2$. Substituting $r = \frac{h}{2}$ and we have $\frac{1}{3}\pi(\frac{h}{2})^2 h = 100$. Simplifying gives $\frac{\pi}{12}h^3 = 100$. Solving for h gives a depth approximately 7.26 cm. I used no calculus, just the formula for the volume of a cone.

   (b) How fast is the level dropping when there are 100 cm^3 of coffee in the funnel? Do you need calculus for this part?

   Solution: As in the previous part, $V = \frac{1}{3}\pi r^2 h$, and $r = \frac{h}{2}$, substituting gives $V = \frac{\pi}{12}h^3$. Differentiating with respect to time t gives

   $$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$

   Now we substitute, $\frac{dV}{dt} = -1.3\ \text{cm}^3/\text{sec}$ and $h = 7.26$ cm.

   $$-1.3 = \frac{\pi}{4} \cdot (7.26)^2 \frac{dh}{dt}$$

   Solving gives $\frac{dh}{dt} = -.031 \frac{\text{cm}}{\text{sec}}$.
   This time I needed calculus to solve the problem, because the cross-sections of the funnel are not constant.

   (c) How long does it take for the funnel to drain? Do you need calculus for this part of the problem?

   Solution: The total volume of the funnel is $V = \frac{1}{3}(4\ \text{cm})^2(8\ \text{cm}) = 134.04\ \text{cm}^3$. The funnel is draining at a constant rate of 1.3 cm^3/sec, so the drain time is approximately 103 seconds. I did not need calculus for this part, because the drain rate is constant.

As in the first problem, it's not realistic that the funnel drains at a constant rate. We will again address this issue using differential equations.

6. Now we'll again use the more accurate model, Toricelli's law, for the rate at which the liquid is draining. Assume the area of the hole in the funnel is $A = 2$ mm^2.

$$\frac{dV}{dt} = -0.6A\sqrt{2gh(t)}$$

(a) Note that $\frac{dV}{dt}$ depends on time, as does $\frac{dh}{dt}$. Show that h satisfies the differential equation $\frac{dh}{dt} = -.68h^{-3/2}$.

Solution: As in the previous part, we have

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

We have that $\frac{dV}{dt} = -0.6 \cdot (.02)\sqrt{2gh} = -.53\sqrt{h}$. This time we do not substitute h since it is varying.

$$-.53\sqrt{h} = \frac{\pi}{4}h^2\frac{dh}{dt}$$
$$\frac{dh}{dt} = -.68h^{-3/2}$$

(b) Solve the differential equation to find h. Use the fact the the funnel is full at time $t = 0$ to solve for the constant.

Solution: Separating variables gives

$$h^{3/2}dh = -.68dt$$

Integrating:

$$\frac{2}{5}h^{5/2} = -.68t + C$$

Now when $t = 0$ we have $h = 8$, so $C = \frac{2}{5}(8)^{5/2} \approx 72.4$.

$$\frac{2}{5}h^{5/2} = -.68t + 72.4$$

Solving gives

$$h = \left(\frac{5}{2}(-.68t + 72.4)\right)^{2/5}$$

(c) How long does it take for the funnel to drain?

Solution: Setting $h = 0$ in the above equations gives

$$-.68t + 72.4 = 0$$

so $t = 106.5$ seconds.