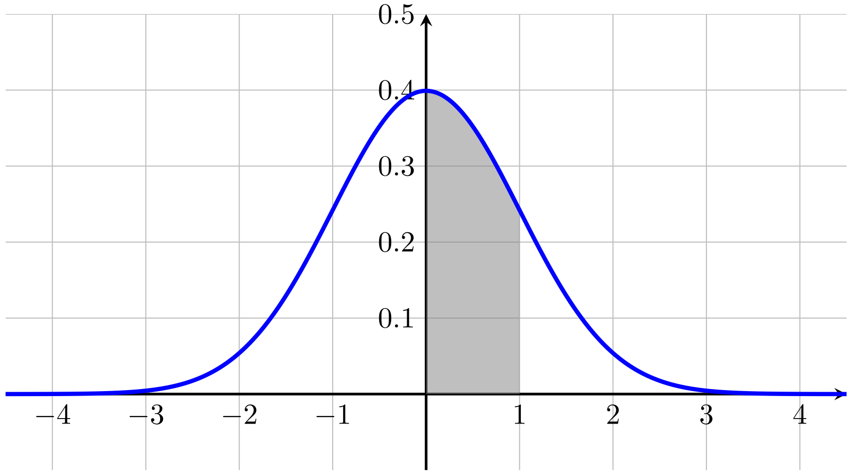
Background for the example:
In probability it is important to be able to find areas under the “bell curve”. The bell curve is formally known as the “normal distribution”, and the function defining the standard normal distribution is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Areas under this curve represent probabilities. For example, say we wanted to know what percentage of students scored between 500 and 600 on the SAT. The mean on the SAT is 500, and the standard deviation is 100, so we are asking for the area under $f(x)$ between 0 and 1.

1. Use technology to draw a graph of $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.



2. On the above graph, shade in the region corresponding to $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$. (If you have taken a probability or statistics class, you will know that this area is approximately 0.34, or 34%.)

Using power series to calculate $\int_0^1 e^{-x^2/2}\,dx$:

Step 1: Write down the power series representation for e^x. On what interval does the series converge? On what interval does the series converge to e^x?

Solution:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We’ll use the ratio test to find the radius of convergence:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 \text{ for all } x.$$

This limit will be 0 regardless of the value of x, thus the interval of convergence is $(-\infty, \infty)$. So the series converges, but this does not show that the Taylor Series necessarily converges to e^x. However, it is shown in the textbook that the series does in fact converge to e^x itself.

Step 2: Make a substitution in the above series to produce a power series representation for $e^{-x^2/2}$. Given your answer in Step 1, on what interval does this series converge to $e^{-x^2/2}$?

Solution:

$$e^{-x^2/2} = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{8 \cdot 3!} + \cdots + \frac{(-x^2/2)^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!}$$

Since the Taylor series for e^x converges to e^x for all x, when we make the given substitution, the new series still converges to the new function for all x.

Step 3: In the integral $\int_0^1 e^{-x^2/2}\,dx$, replace the function with its power series representation from the previous step. Then rewrite the integral of the sum as the sum of the integrals.

Solution:

$$\begin{aligned}\int_0^1 e^{-x^2/2}\,dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!}\,dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-x^2/2)^n}{n!}\,dx \\ &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{2^n \cdot n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} \int_0^1 x^{2n}\,dx\end{aligned}$$

Step 4: Integrate each term. You now have represented the original integral as an infinite series.

Solution:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} \int_0^1 x^{2n} dx &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n!} \cdot \left[\frac{1}{2n+1} - 0\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(2n+1)n!}\end{aligned}$$

Step 5: Now we want to find the sum of the series from Step 4. Notice that it is an alternating series. Use the alternating series remainder estimate to determine how many terms must be added to approximate the sum to within 4 decimal places (Choose n large enough that the error is guaranteed to be less than .00005)

Solution: The alternating series remainder theorem $|S - S_n| < |a_{n+1}|$, where $|S - S_n|$ is our error after summing up to the n term of our series. So, to be accurate to four decimal places we need $\left|\dfrac{(-1)^{n+1}}{2^{n+1}(2(n+1)+1)(n+1)!}\right| < 0.00005$. By trial and error, the smallest value of n that makes the inequality true is $n = 4$. So we need to add up terms of the series from $n = 0$ to $n = 4$ to be accurate to 4 decimal places.

Step 6: Add the number of terms you determined in the previous step to estimate the value of the integral.

Solution:

$$\sum_{n=0}^{4} \frac{(-1)^n}{2^n(2n+1)n!}$$

$$= \frac{(-1)^0}{2^0(2 \cdot 0+1)0!} + \frac{(-1)^1}{2^1(2 \cdot 1+1)1!} + \frac{(-1)^2}{2^2(2 \cdot 2+1)2!} + \frac{(-1)^3}{2^3(2 \cdot 3+1)3!} + \frac{(-1)^4}{2^4(2 \cdot 4+1)4!}$$

$$= 1 + \left(-\frac{1}{6}\right) + \frac{1}{40} + \left(-\frac{1}{336}\right) + \frac{1}{3456} \approx 0.8556$$

Step 7: What is the approximate value of our original integral, $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$?

Solution:

$$\frac{1}{\sqrt{2\pi}} \cdot 0.855624 \approx 0.3413$$

For each of the following series, determine the value of the sum (if it exists).

1. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

   Solution: This series makes me think of $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. I'll substitute $x = 2$ on both sides to get $e^2 = \displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$. The given series converges to e^2.

2. $\displaystyle\sum_{n=0}^{\infty} \frac{5 \cdot 3^n}{(n+1)!}$

   Solution: This series also makes me think of $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. It doesn't match exactly, but close. I'll substitute $x = 3$ to get $e^3 = \displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$. Now I'll take the given series and rewrite it until it looks like the series I know. I'll shift the index by letting $m = n+1$. So $n = m-1$ and when $n = 0$, $m = 1$. I get $\displaystyle\sum_{n=0}^{\infty} \frac{5 \cdot 3^n}{(n+1)!} = 5\sum_{m=1}^{\infty} \frac{3^{m-1}}{m!} = \frac{5}{3}\sum_{m=1}^{\infty} \frac{3^m}{m!} = \frac{5}{3}\left(\sum_{m=0}^{\infty} \frac{3^m}{m!} - 1\right) = \frac{5}{3}(e^3 - 1)$.

3. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$

   Solution: I start with a geometric series, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$. To get the factor of n in the series, I differentiate both sides to get $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=1}^{\infty} nx^{n-1}$. Now I'll substitute $x = \dfrac{1}{2}$, giving $\dfrac{1}{(1-\frac{1}{2})^2} = \displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = 2\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$. I have $4 = 2\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n$, so $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$.

4. $\displaystyle\sum_{n=1}^{\infty} n3^{n-1}$

   Solution: $\displaystyle\lim_{n\to\infty} n3^{n-1} = \infty$, so by the divergence test this series diverges.