Background: We've calculated the Taylor Series centered about $a = 0$ for some important functions. For each of these functions, we know that if it has a power series representation, then it must be actually equal to its Taylor Series on the interval of convergence. In fact, each of the functions below is equal to its Taylor Series, facts that we will show next week. It's worth remembering the Taylor Series for the functions below, in both expanded notation and sigma notation.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{on the interval } (-\infty, \infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{on the interval } (-\infty, \infty)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{on the interval } (-\infty, \infty)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n \quad \text{on the interval } (-1, 1)$$

We will now build new Taylor series from familiar old ones using the techniques of substitution, multipication, differentiation and integration.

1. (a) Write down the Taylor series centered about $a = 0$ for $f(x) = e^x$ in expanded form. Take the derivative. What do you notice? Why does this make sense for $f(x)$?

   Solution:
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$
   Taking the derivative of each term gives:
$$0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$$
   This new series is the same as the series for e^x, which makes sense since the derivative of e^x is e^x.

   (b) Write down the Taylor series centered about $a = 0$ for $f(x) = e^x$ in summation notation (Σ-notation). Take the derivative. Confirm that you get the Taylor series for e^x again.

   Solution: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Taking derivatives with respect to x gives:
$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

2. (a) Write down the Taylor series centered about $a = 0$ for $f(x) = \sin x$ in expanded form. Take the derivative. What do you notice? Why does this make sense for $f(x)$?

Solution:
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + ...$$
Taking the derivative of each term gives: $1 - \frac{x^2}{2} + \frac{x^4}{4!} + ...$
This new series is the same as the series for $\cos x$, which makes sense since $\cos x$ is the derivative of $\sin x$.

(b) Write down the Taylor series centered about $a = 0$ for $f(x) = \cos x$ in summation notation (Σ-notation). Take the derivative. Confirm that you get the Taylor series for the derivative of $\cos x$.

Solution: $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ Taking derivatives of each term gives:
$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right)' = \sum_{n=1}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
Yay, this is the Taylor series for $-\sin x$!

3. Find the Taylor series centered about $a = 0$ for $f(x) = \cos 2x$. Derive the answer working in expanded notation and then again in Σ-notation. Confirm the two answers are the same.

Solution: Begin with the Tayor series for $\cos x$:
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... + \frac{(-1)^n x^{2n}}{(2n)!} + ...$
Now substitute $2x$ in place of x:
$\cos 2x = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - ... + \frac{(-1)^n (2x)^{2n}}{(2n)!} + ...$
In Σ-notation, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, so $\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$

4. Find the Taylor Series centered about $a = 0$ for x^2e^{3x}. Give the answer in Σ-notation.

Solution: Substitute $3x$ in place of x in the Taylor series for e^x, then multiply by x^2:
$x^2e^{3x} = x^2\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} = \sum_{n=0}^{\infty}x^2\frac{3^nx^n}{n!} = \sum_{n=0}^{\infty}\frac{3^nx^{n+2}}{n!}$

5. Find the Taylor Series centered about $a = 0$ for $\sin x^2$.

Solution: Substitute x^2 in the place of x in the Taylor series for $\sin x$: $\sin x^2 = \sum_{n=0}^{\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^nx^{4n+2}}{(2n+1)!}$

6. Find the Taylor Series centered about $a = 0$ for $\frac{e^x - 1}{x}$.

Solution: $\frac{e^x-1}{x} = \frac{-1+1+x+x^2/2+x^3/3!+...}{x} = \frac{x+x^2/2+x^3/3!+...}{x}$
$= 1 + x/2 + x^2/3! + ... = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$

7. Find the Taylor series centered about $a = 0$ for $\dfrac{1}{1-x}$

Solution: This is just a geometric series: $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$

8. Find the Taylor series centered about $a = 0$ for $\dfrac{1}{(1-x)^2}$

Solution: This one is sneaky. Notice that $\left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$. Just differentiate the Taylor series for $\dfrac{1}{1-x}$.

$\left(\displaystyle\sum_{n=0}^{\infty} x^n\right)' = \displaystyle\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots.$

9. Find the Taylor series centered about $a = 0$ for $\dfrac{1}{1+t^2}$

Solution: This time substitute $-t^2$ in place of x in the Taylor series for $\dfrac{1}{1-x}$:

$\dfrac{1}{1+t^2} = \displaystyle\sum_{n=0}^{\infty} (-t^2)^n = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^{2n}$

10. Find the Taylor series centered about $a = 0$ for $\arctan x = \displaystyle\int_0^x \dfrac{1}{1+t^2}\, dt$

Solution: Now we'll integrate the Taylor series we found for $\dfrac{1}{1+t^2}$:

$\displaystyle\int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\, dt$

$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}\Big|_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$